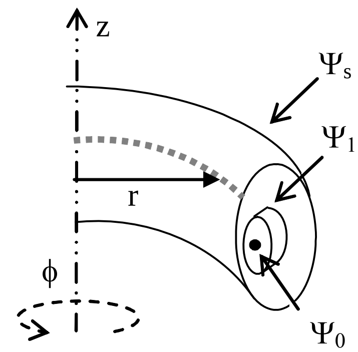
Axisymmetric equilibria from the Grad-Shafranov equation

I. Derivation of the Grad-Shafranov equation



When a straight cylindrical plasma is bent into a torus, the magnetic field lines map out flux surfaces that are often labelled by a value of a stream function ψ.
For toroidal plasma there is axisymmetry, meaning $d/d\phi = 0$, where ϕ is the toroidal angle in the cylindrical coordinate system r,ϕ,z. This symmetry can be used to simplify the force balance equation so that solutions can be more easily obtained. As in the case of the straight cylinder, the current profile and the pressure profile must be specified. The fields B_r and B_z are quantities that come out of the calculation as well as the applied toroidal field B_ϕ that has been modified by the plasma current and pressure.

The vector equilibrium equation is:
where P is a scalar pressure.

$$\vec{J}\times\vec{B}-\vec{\nabla}P=0$$

Dot products with J and B show that:

$$\vec{J}\cdot\vec{\nabla}P=0 \qquad \vec{B}\cdot\vec{\nabla}P=0$$

which indicates that the pressure gradient is perpendicular to both **B** and **J**.

Ampere's law is:

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$$

The magnetic vector potential is defined as:

$$\vec{\nabla}\times\vec{A}=\vec{B}$$

Application of Stokes' theorem gives:

$$\oint\vec{B}\cdot d\vec{l}=\mu_0\int\vec{J}\cdot d\vec{S}=\mu_0 I \qquad \text{and} \qquad \oint\vec{A}\cdot d\vec{l}=\int\vec{B}\cdot d\vec{S}=\Phi=2\pi\psi$$

where I is the current through surface S, Φ is the magnetic flux through surface S, and ψ is a stream function that is equal to the magnetic flux divided by 2π. If the two integrals above with dl are both performed around a circle of radius r, then

$$B_\phi=\frac{\mu_0}{2\pi r}I \qquad A_\phi=\frac{1}{2\pi r}\Phi_\phi=\frac{\psi}{r}$$

where I is the current enclosed by the contour and $2\pi\psi$ is the flux enclosed by the contour.
B_r and B_z are simply related to ψ:

$$B_r=\left(\vec{\nabla}\times\vec{A}\right)_r=-\frac{\partial}{\partial z}A_\phi=-\frac{1}{r}\frac{\partial}{\partial z}\psi \qquad B_z=\left(\vec{\nabla}\times\vec{A}\right)_z=\frac{1}{r}\frac{\partial}{\partial r}rA_\phi=\frac{1}{r}\frac{\partial}{\partial r}\psi$$

I is constant on surfaces of constant ψ because **J** is locally parallel to these surfaces.

Stokes' theorem applied above indicates that **B** is to $2\pi\psi$ as **J** is to I. In the line above, replacing **B** by **J** and ψ by $I/2\pi$ we obtain:

$$J_r = -\frac{1}{2\pi r}\frac{\partial}{\partial z}I \qquad J_z = \frac{1}{2\pi r}\frac{\partial}{\partial r}I$$

Try it: Convince yourself that if the contour (at r in the figure on the previous page) is shifted upward by an amount dz, then I changes by $-2\pi r J_r$.

It is interesting to note that **B** can be written: $\vec{B} = (\vec{\nabla}\times\vec{A})_\phi + B_\phi \vec{e}_\phi$
where e denotes a unit vector.

Ampere's law becomes: $$(\vec{\nabla}\times\vec{B})_\phi = \mu_0 J_\phi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{\partial}{\partial z}\frac{1}{r}\frac{\partial}{\partial z}\psi - \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}\psi$$

which gives: $$\mu_0 r J_\phi = -\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right]\psi = -L(\psi)$$ where L is the bracketed operator.

$\vec{J}\times\vec{B} - \vec{\nabla}P = 0$ becomes

$$0 = J_\phi B_z - J_z B_\phi - \frac{\partial P}{\partial r} = J_\phi \frac{1}{r}\frac{\partial}{\partial r}\psi - \left(\frac{1}{2\pi r}\frac{\partial}{\partial r}I\right)\left(\frac{\mu_0 I}{2\pi\, r}\right) - \frac{\partial P}{\partial r}$$

J_ϕ is found by multiplying through by $r\, dr/d\psi$ to obtain:

$$J_\phi = r\frac{\partial r}{\partial\psi}\frac{\partial P}{\partial r} + \left(\frac{\partial r}{\partial\psi}\frac{1}{2\pi}\frac{\partial}{\partial r}I\right)\left(\frac{\mu_0 I}{2\pi r}\right) = r\frac{\partial P}{\partial\psi} + \left(\frac{\mu_0 I}{4\pi^2 r}\right)\left(\frac{\partial I}{\partial\psi}\right)$$ thus

$$\mu_0 r J_\phi = \mu_0 r^2\frac{\partial P}{\partial\psi} + \frac{\mu_0^{\,2} I}{4\pi^2}\left(\frac{\partial I}{\partial\psi}\right) = -L(\psi)$$

These are our results which are two relationships that must be satisfied.
The first of these is the Grad-Shafranov equation:

$$\left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right]\psi + \mu_0 r^2\frac{\partial P}{\partial\psi} + \frac{\mu_0^{\,2}}{8\pi^2}\left(\frac{\partial(I^2)}{\partial\psi}\right) = 0$$ **The Grad-Shafranov Equation**

The second result is that these terms are minus and plus (respectively) $\mu_0 r J_{\phi 0}$.

The second relation will be used to place constraints the functions $P(\psi)$ and $I^2(\psi)$.

The similarity of the operator $L(\psi)$ to the Laplacian operator in the cylindrical version of Poisson's equation suggests that we can find a solution to the Grad-Shafranov equation by relaxation, as was done for problems in electrostatics.

In summary:
B_r, B_z and J_ϕ are determined by the $\psi(r,z)$ that we find.
J_r and J_z are functions of $I(\psi)$ that we can find as a function of r and z after $\psi(r,z)$ is found.
$P(\psi)$ and $I^2(\psi)$ are functions that we specify.

II. A relaxation method for the Grad Shafranov equation

A. Finite-difference form for $L(\psi)$.

In the pages that follow, we will assume (following reference 1) that the functions P and I^2 vary as ψ^2 making their derivatives linear in ψ:

$$\alpha r^2\psi = \mu_0 r^2 \frac{\partial P}{\partial \psi} \qquad \beta\psi = \frac{\mu_0{}^2}{8\pi^2}\left(\frac{\partial (I^2)}{\partial \psi}\right)$$ where α and β are constants.

The Grad-Shafranov equation with these simpified choices for $P(\psi)$ and $I^2(\psi)$ is:

$$L(\psi) + (\alpha r^2 + \beta)\psi = 0 \qquad \text{where} \qquad L(\psi) = \left[r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial}{\partial z^2}\right]\psi$$

and where ψ is a function of r and z.

Let $\psi_{i,j}$ be shorthand notation for ψ defined on a grid r_i, z_j.
With $L(\psi)$ in finite-difference form, the Grad-Shafranov equation becomes:

$$\frac{r_i}{(\Delta r)^2}\left[\frac{\psi_{i+1,j}}{r_{i+\frac{1}{2}}} - \left(\frac{1}{r_{i+\frac{1}{2}}} + \frac{1}{r_{i-\frac{1}{2}}}\right)\psi_{i,j} + \frac{\psi_{i-1,j}}{r_{i-\frac{1}{2}}}\right] + \frac{1}{(\Delta z)^2}\left[\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}\right] + (\alpha r^2 + \beta)\psi_{i,j} = 0$$

where $r_{i+1/2} = r_i + 0.5\,\Delta r$. This equation can be multiplied by Δr^2 and rearranged to give:

$$\left(\frac{r_i}{r_{i+\frac{1}{2}}} + \frac{r_i}{r_{i-\frac{1}{2}}} + \frac{2(\Delta r)^2}{(\Delta z)^2}\right)\psi_{i,j} - (\Delta r)^2(\alpha r^2 + \beta)\psi_{i,j} = r_i\left[\frac{\psi_{i+1,j}}{r_{i+\frac{1}{2}}} + \frac{\psi_{i-1,j}}{r_{i-\frac{1}{2}}}\right] + \frac{(\Delta r)^2}{(\Delta z)^2}\left[\psi_{i,j+1} + \psi_{i,j-1}\right]$$

Further rearrangement gives a formula for finding ψ by relaxation:

$$\psi_{i,j} \leftarrow \frac{r_i\left(\frac{\psi_{i+1,j}}{r_{i+\frac{1}{2}}} + \frac{\psi_{i-1,j}}{r_{i-\frac{1}{2}}}\right) + \frac{(\Delta r)^2}{(\Delta z)^2}\left[\psi_{i,j+1} + \psi_{i,j-1}\right]}{\left(\frac{r_i}{r_{i+\frac{1}{2}}} + \frac{r_i}{r_{i-\frac{1}{2}}} + \frac{2(\Delta r)^2}{(\Delta z)^2}\right) - (\Delta r)^2(\alpha r^2 + \beta)}$$

where the left arrow indicates that the value on the left is to be replaced by the value on the right. This prescription is repeated until ψ has relaxed to its final form.
If $\alpha = \beta = 0$, the new $\psi_{i,j}$ is simply a weighted average of the four neighboring values.

B. The grids and the size of the device:

The domain will be a rectangular grid which is a cross section of the plasma. If we wish to have a circular plasma we can inscribe a circle within a square cross section.

The grids will be r_i and z_j. We will let the range of r be 1 to 2 meters and the range of z be -0.5 to +0.5 meters. Note that r_{max} is a variable name and r_i is a subscripted variable.

$r_{min} := 1.0 \quad r_{max} := 2.0 \quad imax := 10 \quad \Delta r := \frac{r_{max} - r_{min}}{imax} \quad i := 0 .. imax \quad r_i := r_{min} + i \cdot \Delta r$

$z_{max} := 0.5 \quad z_{min} := -z_{max} \quad jmax := 10 \quad \Delta z := \frac{z_{max} - z_{min}}{jmax} \quad j := 0 .. jmax \quad z_j := z_{min} + j \cdot \Delta z$

$r^T =$

	0	1	2	3	4	5	6	7	8	9
0	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	...

$z^T =$

	0	1	2	3	4	5	6	7	8	9
0	-0.5	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	...

Define the j value that is on the midplane where z = 0 $\quad jmid := 0.5 \cdot jmax$

Define a major radius that is approximately the location of the minor axis: $\quad r_{major} := 0.5 \cdot (\min(r) + \max(r)) \quad r_{major} = 1.5$

C. The applied magnetic field $B_{\phi 0}$ and the plasma current density $J_{\phi 0}$:

We will specify several quantities at the magnetic axis, labelled ψ_0 in the figure.
The experimenter chooses the toroidal field to apply and the plasma current.

We will assume that the field $B_{\phi 0}$ = 1 T near the magnetic axis: $\quad B_{\phi 0} := 1.0$ Tesla

We will choose the current density to give a safety factor q_0 of 2.0 on the magnetic axis because this is a currrent density that satisfies a stability requirement $q_0 > 1$. $\quad q_0 := 2$

Highlighted values are two the the three "knobs" that can be turned by the experimenter.

The definition of q at a small distance a from the magnetic axis is:

$$q_0 = \frac{B_{\phi 0} a}{B_{pol} r_{major}}$$

where a is the minor radius variable in toroidal coordinates.

The field B_{pol} is the poloioidal field that encircles the magnetic axis.

For Ampere's law we need the constant: $\quad \mu_0 := 4 \cdot \pi \cdot 10^{-7}$

Ampere's law near the magnetic axis gives:

$$2\pi a B_{pol} = \pi a^2 \mu_0 J_{\phi 0}$$

The equations above combined give:

$$J_{\phi 0} = \frac{2B_{pol}}{a\mu_0} = \frac{2B_{\phi 0}}{q_0 \mu_0 r_{major}}$$

The current density on axis is then:

$$J_{\phi 0} := \frac{2 \cdot B_{\phi 0}}{q_0 \cdot \mu_0 \cdot r_{major}} \qquad J_{\phi 0} = 5.305 \times 10^5 \ \text{A/m}^2$$

A more useful quantity is $\mu_0 r J_\phi$, and in Mathcad this will be written as one word μrJ.

$$\mu_0 r J_\phi = \frac{2B_{\phi 0}}{q_0} \qquad \mu rJ := \frac{2 \cdot B_{\phi 0}}{q_0}$$

Recall that this quantity is –L(ψ) in the Grad-Shafranov equation.

$\mu rJ = 1.000$ The resulting value of the current density parameter

We will not know where the magnetic axis is located until we have a solution. In the lines above, when we needed a value for r at the magnetic axis where $\psi = \psi_0$, we used the geometric center of the plasma as an approximation for the radial location of ψ_0.

D. The initial guess for the function ψ(r,z),

For our relaxation method, we must have a starting function that is near to the desired function. We can assign ψ_s (the value at the plasma surface) to be zero without loss of generality.

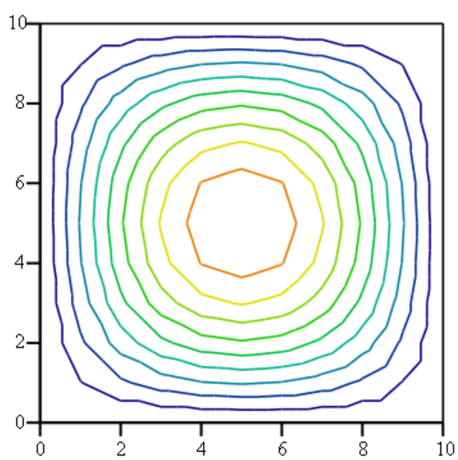
We will guess that the sine function is near the lowest order mode (eigenfunction) of the equation.

The starting guess for ψ on the magnetic axis: $\psi_0 := 1.0$

Starting guess for ψ(r,z):

$$\psi_{trial}(r,z) := \psi_0 \cdot \sin\left[\left(\frac{r_{max} - r}{r_{max} - r_{min}}\right) \cdot \pi\right] \cdot \sin\left(\frac{z_{max} - z}{z_{max} - z_{min}} \cdot \pi\right)$$

These values will be put in a matrix for plotting: $\psi_{i,j} := \psi_{trial}(r_i, z_j)$



ψ

The trial function generates the flux surfaces we expect to see but the amplitude of the function ψ(r,z) has been chosen arbitrarily. On the magnetic axis, the function must satisfy:

$$\mu_0 r J_{\phi 0} = -L(\psi)$$

From the symmetry, we can safely assume that the magnetic axis occurs at the midplane where j = jmid, so we only need to evaluate L at the midplane and search for the maximum value. This maximum should be $\mu_0 r J \phi_0$ and will be rescaled to this value later.

Now we will find a better value for ψ_0. We will evaluate L(ψ) at ψ_0 then adjust the amplitude of ψ so that L(ψ) has the desired value. L cannot be defined at the boundary points by finite-differencing, so we will only find L at interior points. L(ψ) will be made a function in a program loop so that it doesn't have to be written out each time it appears.

$$L(\psi) := \left| \begin{array}{l} L_{imax} \leftarrow 0 \\ \text{for } ii \in 1 .. imax - 1 \\ \quad L_{ii} \leftarrow \frac{r_{ii}}{\Delta r^2} \cdot \left[\frac{\psi_{ii+1,jmid}}{r_{ii} + \frac{\Delta r}{2}} - \left(\frac{1}{r_{ii} + \frac{\Delta r}{2}} + \frac{1}{r_{ii} - \frac{\Delta r}{2}} \right) \cdot \psi_{ii,jmid} + \frac{\psi_{ii-1,jmid}}{r_{ii} - \frac{\Delta r}{2}} \right] \ldots \\ \qquad + \frac{1}{\Delta z^2} \cdot \left(\psi_{ii,jmid+1} - 2 \cdot \psi_{ii,jmid} + \psi_{ii,jmid-1} \right) \\ L \end{array} \right.$$

The largest value of –L(ψ) is the value on the magnetic axis and this value should be μrJ.

ψ_{trial} gives the following for μrJ at ψ_0: $\max(-L(\psi)) = 19.588$

The value that we need is: $\mu rJ = 1.000$

We will rescale the value of ψ_0 so that –L(ψ) produces the desired μrJ.

$$\text{ScaleFactor} := \frac{\mu rJ}{\max(-L(\psi))} \qquad \text{ScaleFactor} = 0.051$$

Rescale ψ_0: $\psi_0 := \psi_0 \cdot \text{ScaleFactor}$ $\qquad \psi_0 = 0.051$ $\qquad$ The new ψ_0.

E. The function P(ψ) and the parameter α

Recall that we assumed the P(ψ) varied as ψ^2 and defined: $\alpha r^2 \psi = \mu_0 r^2 \frac{\partial P}{\partial \psi}$

Following reference 1 we will define $P(\psi) = P_0 \left(\frac{\psi}{\psi_0} \right)^2$ thus $\frac{\partial P}{\partial \psi} = 2P_0 \frac{\psi}{\psi_0^2}$

where P_0 is the pressure on the magnetic axis.
Assume that the plasma pressure is 1% of the toroidal field pressure. Then P_0 is

$P_0 := 0.01 \cdot \left(\frac{B_{\phi 0}{}^2}{2 \cdot \mu_0} \right)$ The plasma pressure is the third of the three "knobs" that we can adjust. In this formula we have ignored modification to $B_{\phi 0}$ by the plasma.

Then $P_0 = 3.979 \times 10^3$ Pascals and $\alpha := \frac{2 \cdot \mu_0 \cdot P_0}{\psi_0{}^2}$ $\qquad \alpha = 3.837$

ψ_0 changes during the relaxation process, thus α must be updated continually if P_0 is to be preserved. .

F. The current function $I(\psi)$ and the parameter β

The current in the toridal field coils flows through conductors in the "doughnut hole" and this current determines the value of $I(\psi)$ at the plasma surface ψ_s (see the figure on the first page).

Following reference 1, we will define:

$$I^2(\psi) = I_s^2 + \left(I_0^2 - I_s^2\right)\left(\frac{\psi}{\psi_0}\right)^2$$

where I_s is the value of I at the plasma surface and I_0 is the value at the magnetic axis.

From the definition: $$\beta\psi = \frac{\mu_0{}^2}{8\pi^2}\left(\frac{\partial\left(I^2\right)}{\partial\psi}\right)$$ we find: $$\beta = \left(\frac{\mu_0{}^2}{4\pi^2}\right)\frac{\left(I_0^2 - I_s^2\right)}{\psi_0^2}$$

The value of β must satisfy at ψ_0: $$\mu_0 r J_{\phi 0} = \left(\alpha r^2 + \beta\right)\psi_0 = -L(\psi_0)$$

We have assigned a value to α and to μrJ, hence we can solve for β using values at ψ_0: $$\beta := \frac{\mu rJ}{\psi_0} - 2\cdot\mu_0\cdot\frac{r_{major}{}^2\cdot P_0}{\psi_0{}^2}$$

This is the value for β: $\beta = 10.955$

ψ_0 changes during the relaxation process, thus β must be updated continually in order to prevent μrJ from changing during the relaxation.

The current function at the plasma surface is:
This value arises from the current in the torodal field coils and can be determined from the vacuum (no plasma) magnetic field.

$$I_s := \frac{2\cdot\pi\cdot r_{major}\cdot B_{\phi 0}}{\mu_0}$$

$I_s = 7.5 \times 10^6$ Amps

The parameter I_0 must be defined so that so that μrJ has the correct value on the magnetic axis. We can solve for I_0 using:

$$I_0 := \sqrt{\frac{4\cdot\pi^2}{\mu_0{}^2}\cdot\left(\mu rJ - 2\cdot\mu_0\cdot r_{major}{}^2\cdot\frac{P_0}{\psi_0}\right) + I_s{}^2}$$

We find: $I_0 = 8.38 \times 10^6$ Amps

G. Testing the trial solution

The relaxation method works best if a trial solution is provided that is near to the final solution. We will test our trial solution by plotting the two terms of the Grad-Shafranov equation to see that they nearly balance one another.

Redefine ψ(r,z) with the new ψ_0:

$$\psi_{trial}(r,z) := \psi_0 \cdot \sin\left[\left(\frac{r_{max} - r}{r_{max} - r_{min}}\right)\cdot\pi\right]\cdot\sin\left(\frac{z_{max} - z}{z_{max} - z_{min}}\cdot\pi\right)$$

Redefine the matrix ψ:

$$\psi_{i,j} := \psi_{trial}\left(r_i, z_j\right)$$

$$Term\_1 := -L(\psi) \qquad Term\_2_i := \left[\alpha\cdot\left(r_i\right)^2 + \beta\right]\cdot\psi_{i,jmid}$$

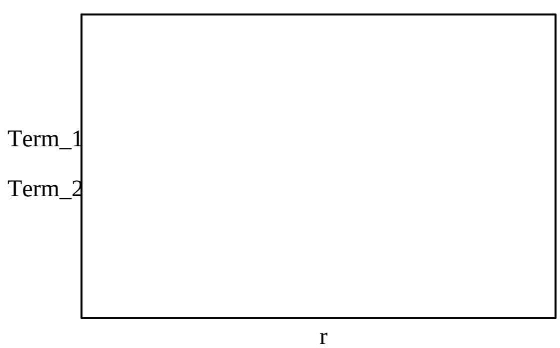
Term_1 =

	0
0	0
1	0.446
2	0.695
3	0.881
4	0.986
5	1
6	0.92
7	0.754
8	0.517
9	0.23
10	0

Term_2 =

	0
0	0
1	0.246
2	0.495
3	0.72
4	0.897
5	1
6	1.009
7	0.91
8	0.702
9	0.391
10	0

Plot of the trial solution:



The trial solution appears to be near to a solution to the equation. The relaxation method should generate a final ψ(r,z) such that these two curves are coincident.

III. The program loop

This is a program loop that solves our problem by successive approximations:

$iters := 3\cdot imax\cdot jmax$ The number of iterations should be several times imax * jmax.

Grad Shafranov equation parameters: $\mu rJ = 1$ $\alpha = 3.837$ $\beta = 10.955$ $\psi_0 = 0.051$

$$
\begin{aligned}
Psi := \; & Psi_{imax,jmax} \leftarrow 0 \\
& \psi temp_{imax,jmax} \leftarrow 0 \\
& Psi \leftarrow \psi \\
& New\beta \leftarrow \beta \\
& New\alpha \leftarrow \alpha \\
& \text{for } k \in 1..iters \\
& \quad \text{for } i \in 1..imax-1 \\
& \quad\quad \text{for } j \in 1..jmax-1 \\
& \quad\quad\quad \psi temp_{i,j} \leftarrow \frac{r_i\cdot\left(\frac{Psi_{i-1,j}}{r_i-0.5\cdot\Delta r}+\frac{Psi_{i+1,j}}{r_i+0.5\cdot\Delta r}\right)+\frac{\Delta r^2}{\Delta z^2}\cdot\left(Psi_{i,j+1}+Psi_{i,j-1}\right)}{\left[\frac{r_i}{r_i-0.5\cdot\Delta r}+\frac{r_i}{r_i+0.5\cdot\Delta r}+2\left(\frac{\Delta r}{\Delta z}\right)^2\right]-\Delta r^2\cdot\left[New\alpha\cdot\left(r_i\right)^2+New\beta\right]} \\
& \quad PsiMax \leftarrow \max(\psi temp) \\
& \quad iaxis \leftarrow match\left(PsiMax, \psi temp^{\langle jmid\rangle}\right)_0 \\
& \quad New\alpha \leftarrow 2\cdot\mu_0\cdot\frac{P_0}{PsiMax^2} \\
& \quad ScaleFactor \leftarrow \frac{\mu_0\cdot r_{iaxis}\cdot J_{\phi 0}}{\max(-L(\psi temp))} \\
& \quad New\beta \leftarrow \frac{\mu_0\cdot r_{iaxis}\cdot J_{\phi 0}}{PsiMax} - New\alpha\cdot\left(r_{iaxis}\right)^2 \\
& \quad Psi \leftarrow \psi temp \\
& Psi_{0,0} \leftarrow ScaleFactor \\
& Psi_{1,0} \leftarrow New\beta \\
& Psi
\end{aligned}
$$

In the first few lines Psi and ψtemp are initialized to zero. The answer matrix Psi is given the starting value ψ. In the relaxation, ScaleFactor is used to adjust Psi so that the scaled values give μrJ = -L(Psi). After each relaxation, Newα is redefined using the new PsiMax and Newβ is redefined so that $\mu rJ = (\alpha\, r^2+\beta)$ Psi. In the final lines, we have taken the parameters ScaleFactor and Newβ and put them into "unused" places in the matrix Psi so that we can observe their values after the loop has stopped.

Note that PsiMax occurs at the radius r_{iaxis}.

Now look at the ouptut:

$ScaleFactor := Psi_{0,0}$ $ScaleFactor = 1$ This factor being near unity shows that ψ is changing very little at the end of the program loop.

$New\beta := Psi_{1,0}$ $New\beta = 11.066$ This is the value of β that is consistent with the final "relaxed" value of ψ_0.

$Psi_{0,0} := 0$ $Psi_{1,0} := 0$ Reset these places in the matrix to their original values (they are on the boundary).

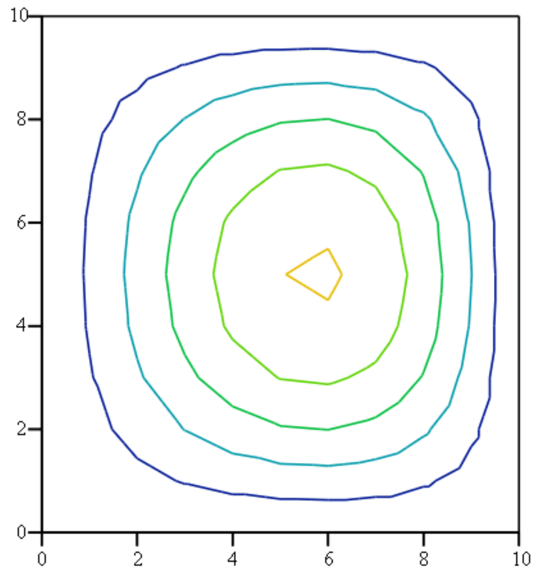
IV. The solutions for ψ(r,z), J_r, J_z, B_r, B_z, J_ϕ etc.

A. The solution for ψ

$Psi^T =$

	0	1	2	3	4	5	6	7	8	9
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.000	0.004	0.007	0.011	0.014	0.015	0.016	0.015	0.011	0.006
2	0.000	0.007	0.014	0.020	0.026	0.029	0.030	0.028	0.021	0.012
3	0.000	0.009	0.019	0.028	0.035	0.040	0.041	0.038	0.029	0.016
4	0.000	0.011	0.022	0.033	0.042	0.047	0.049	0.045	0.035	0.019
5	0.000	0.011	0.023	0.034	0.044	0.050	0.051	0.047	0.036	0.020
6	0.000	0.011	0.022	0.033	0.042	0.047	0.049	0.045	0.035	0.019
7	0.000	0.009	0.019	0.028	0.035	0.040	0.041	0.038	0.029	0.016
8	0.000	0.007	0.014	0.020	0.026	0.029	0.030	0.028	0.021	0.012
9	0.000	0.004	0.007	0.011	0.014	0.015	0.016	0.015	0.011	0.006
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	...

The plotted answer matrix for ψ(r,z):



Psi

The answer matrix gives us new values for two parameters that we initially guessed:

Old value: $\psi_0 = 0.05105$

New value:

$\psi_0 := \max(\mathrm{Psi})$ $\psi_0 = 0.05125$

Old value: $\alpha = 3.837$

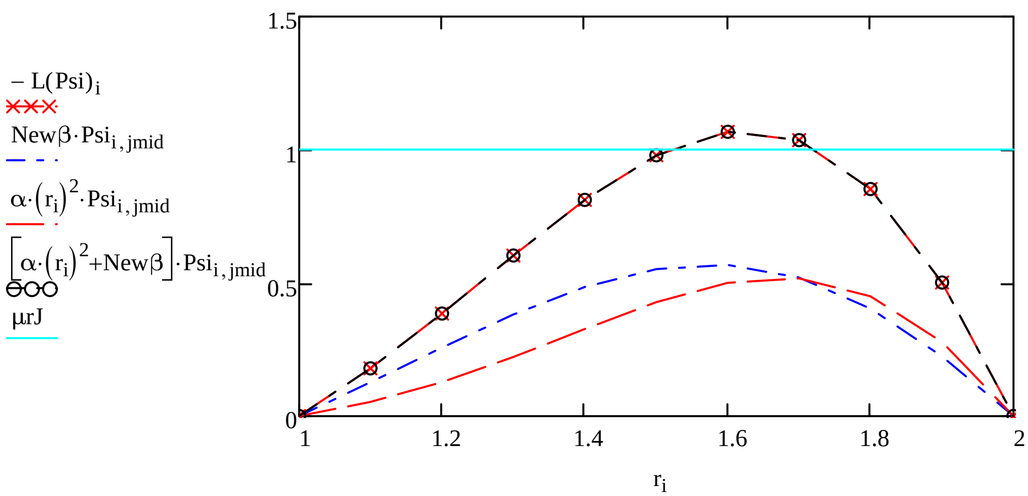
New value:

$\alpha := 2 \cdot \mu_0 \cdot \frac{P_0}{\psi_0^2}$ $\alpha = 3.808$

Shafranov shift:
Note that the magnetic axis is slightly shifted to the outside of the torus. This shift increases with plasma pressure and is named the Shafranov shift.

Is the equation solved?

We can verify that we have a solution by plotting the two terms in the Grad-Shafranov equation (x's and 0's in the graph) and showing that they are equal in magnitude. For reference, $\alpha r^2 \psi$ and $\beta\psi$ are also plotted.



The plot shows that indeed $-\mathrm{L}(\psi) = (\alpha r^2 + \beta)\psi$

B. The solutions for B_r and B_z

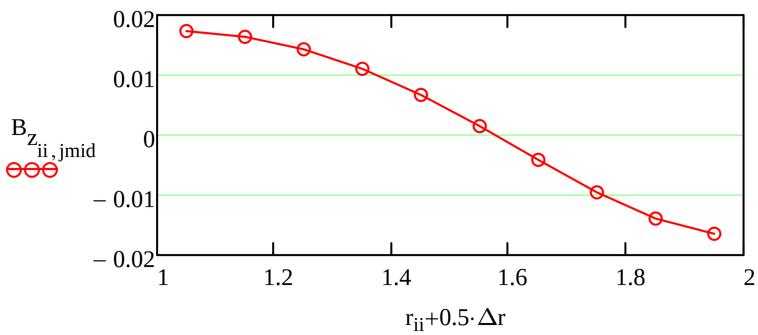
$$B_r = -\frac{1}{r}\frac{\partial}{\partial z}\psi \qquad B_z = \frac{1}{r}\frac{\partial}{\partial r}\psi$$

In finite-difference form these are: $ii := 0 .. imax - 1$ $jj := 0 .. jmax - 1$

$$B_{r_{ii,jj}} := -\frac{1}{2\cdot\pi\cdot\left(r_{ii} + 0.5\cdot\Delta r\right)}\cdot\frac{Psi_{ii,jj+1} - Psi_{ii,jj}}{\Delta z} \qquad B_{z_{ii,jj}} := \frac{1}{2\cdot\pi\cdot\left(r_{ii} + 0.5\cdot\Delta r\right)}\cdot\frac{Psi_{ii+1,jj} - Psi_{ii,jj}}{\Delta r}$$

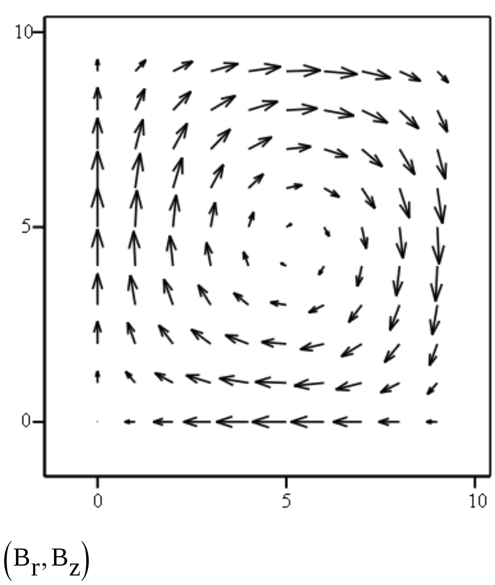
Note that the finite-differences are centered half way between grid points ri.

Plot of the field B_z at the midplane of the device:



The simple finite differencing of Psi above produces derivative values centered between grid points r_i.

Vector plot of B projected onto the r,z plane:



Note that the magnetic field lines encircle the magnetic axis.
Mathcad labels the axes with the subscripts of the matrix thus we cannot label these axes with r and z values.

C. The value of the Shafranov shift:

The magnetic axis can be found from the zero-crossing of B_z. We will find an accurate value by interpolation using the root function. We must first convert the array of B_z values on the midplane to a continuous function using splines:

$Bzmidplane_{ii} := Bz_{ii,jmid}$ Vector of values at the midplane.

$rr_{ii} := r_{ii} + 0.5 \cdot \Delta r$ r values at the half-grid points where B_z was calculated.

$vs := cspline(rr, Bzmidplane)$ vs is the vector of spline coefficients used by interp.

$Bz(x) := interp(vs, rr, Bzmidplane, x)$ Bz is the continuous function made by interpolation.

$r_{axis} := root(Bz(x), x, r_{min}, r_{max})$ The magnetic axis from the root finder.

The difference between the geometric axis and the magnetic axis is the Shafranov shift:

$r_{axis} - r_{major} = 0.076$ meters The Shafranov shift.

Recall that: $r_{axis} = 1.576$

Try it: Change the plasma pressure and observe that change in the Shafranov shift. Be sure to check that the relaxation scheme has found a solution.

D. The solution for J_ϕ

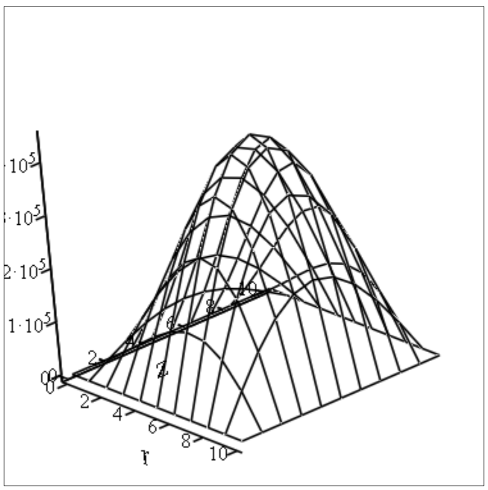
Recall that $\mu r J_\phi$ = -L(ψ). Use the program loop below to find L(ψ) for all r and z.

$$J\phi(\psi) := \left| \begin{array}{l} L2_{imax,jmax} \leftarrow 0 \\ \text{for } ii \in 1..imax-1 \\ \quad \text{for } jj \in 1..jmax-1 \\ \quad\quad L2_{ii,jj} \leftarrow \frac{r_{ii}}{\Delta r^2} \cdot \left[\frac{\psi_{ii+1,jj}}{r_{ii}+\frac{\Delta r}{2}} - \left(\frac{1}{r_{ii}+\frac{\Delta r}{2}} + \frac{1}{r_{ii}-\frac{\Delta r}{2}} \right) \cdot \psi_{ii,jj} + \frac{\psi_{ii-1,jj}}{r_{ii}-\frac{\Delta r}{2}} \right] \ldots \\ \quad\quad\quad + \frac{1}{\Delta z^2} \cdot \left(\psi_{ii,jj+1} - 2 \cdot \psi_{ii,jj} + \psi_{ii,jj-1} \right) \\ J \leftarrow \frac{-L2}{\mu_0 \cdot r_{ii}} \\ J \end{array} \right.$$

$\frac{J\phi(Psi)}{10^5} =$

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0.23	0.44	0.61	0.71	0.75	0.71	0.61	0.44	0.23	0
2	0	0.5	0.95	1.3	1.53	1.61	1.53	1.3	0.95	0.5	0
3	0	0.78	1.48	2.04	2.4	2.52	2.4	2.04	1.48	0.78	0
4	0	1.05	2	2.75	3.23	3.4	3.23	2.75	2	1.05	...

Surface plot of the current density J_ϕ:



$J\phi(Psi)$

L(ψ) must go to zero at the boundary because the derivatives of I(ψ) and P(ψ) go to zero at the boundary. Thus we can assign zero to the values on the boundary without doing any calculation.

E. Solutions for J_r and J_z

The formulas are:

$$J_z = \frac{1}{2\pi r}\frac{d}{dr}I = \frac{1}{2\pi r}\frac{d\psi}{dr}\frac{\partial}{\partial\psi}I \qquad J_r = -\frac{1}{2\pi r}\frac{d}{dz}I = -\frac{1}{2\pi r}\frac{d\psi}{dz}\frac{\partial}{\partial\psi}I$$

Define the current function I(ψ) and define its first derivative:

$$I(\Psi) := \sqrt{I_s^{\,2} + \left(\frac{4\cdot\pi^2}{\mu_0^{\,2}}\right)\cdot\beta\cdot\Psi^2} \qquad dI(\Psi) := \frac{d}{d\Psi}I(\Psi)$$

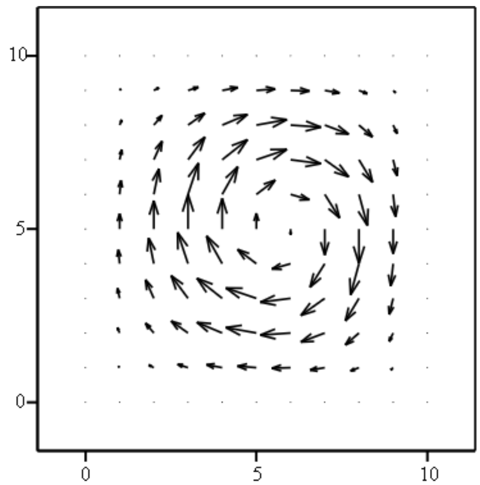
Initialize J_r and J_z matrices: $Jz_{i,j} := 0 \quad Jr_{i,j} := 0$

Find J_r and J_z at the interior points by finite-differencing: $ii := 1 .. imax - 1 \qquad jj := 1 .. jmax - 1$

$$Jz_{ii,j} := \frac{1}{2\cdot\pi\cdot r_{ii}}\cdot\left(\frac{Psi_{ii+1,j} - Psi_{ii-1,j}}{2\cdot\Delta r}\right)\cdot dI\left(Psi_{ii,j}\right) \quad Jr_{i,jj} := -\left[\frac{1}{2\cdot\pi\cdot r_i}\cdot\left(\frac{Psi_{i,jj+1} - Psi_{i,jj-1}}{2\cdot\Delta r}\right)\cdot dI\left(Psi_{i,jj}\right)\right]$$

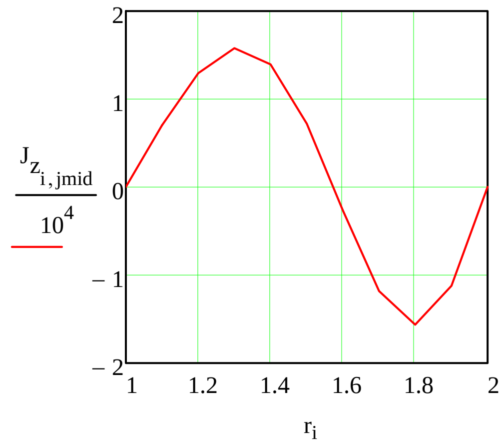
Note that these finite-differences are centered at grid points. The boundary points were initialized to zero and were not calculated because $I(\psi)$ was chosen so that these values would be zero.

Vector field plot of Jr and Jz:



(J_r, J_z)

Plot of Jz as a function of radius at the midplane:



F. Solution for the toroidal field B_ϕ:

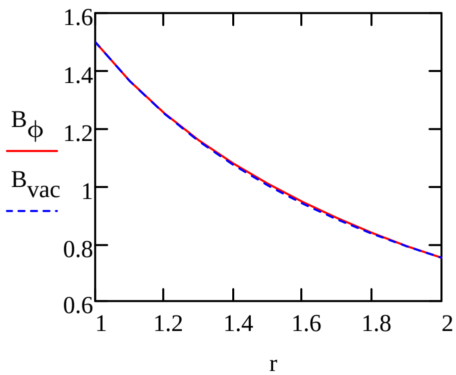
The toroidal field is found from the current function I: $B_\phi(r,z) = \frac{\mu_0}{2\pi r} I(\psi(r,z))$

The function $I(\psi)$ is defined on the previous page. We will plot B_ϕ at the midplane where z = 0.

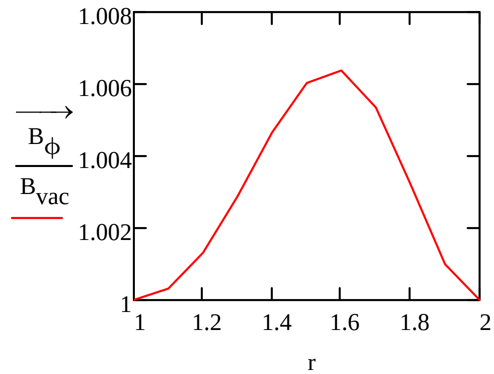
The field modified by the plasma is: $$B\phi_i := \frac{\mu_0 \cdot I(Psi_{i,jmid})}{2 \cdot \pi \cdot r_i}$$

The vacuum field is the field with $\psi = 0$: $$Bvac_i := \frac{\mu_0 \cdot I(0)}{2 \cdot \pi \cdot r_i}$$

The toroidal field shows only a small modification:



The ratio of the fields shows whether the plasma is diamagnetic or paramagnetic:

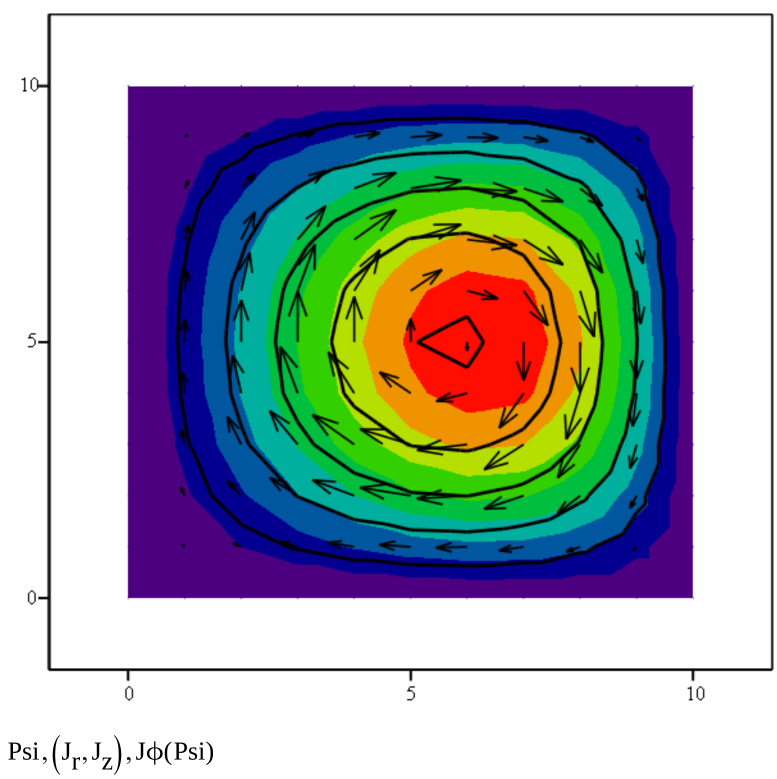


Try it: Find the value of the plasma pressure P_0 where the perturbation to the toroidal field changes from paramagnetic (B_ϕ increased on axis) to diamagnetic (B_ϕ decreased on axis).

G. More sophisticated plotting

It is possible in Mathcad to put several kinds of plots on one set of axes.

In the plot to the right the potential Psi is plotted as black contours, the currents J_r and J_z are plotted as vectors, and the toroidal current density J_ϕ is plotted as filled contours.



The plot above is easily combined with its mirror image to show a vertical slice through the tokamak plasma. This is done by first creating mirror images of the data sets using the reverse command. Second, a set of zeroes is created to fill in the "donut hole." Last, the stack command is used to create a larger data matrix with the left side, the "donut hole," and the right side.

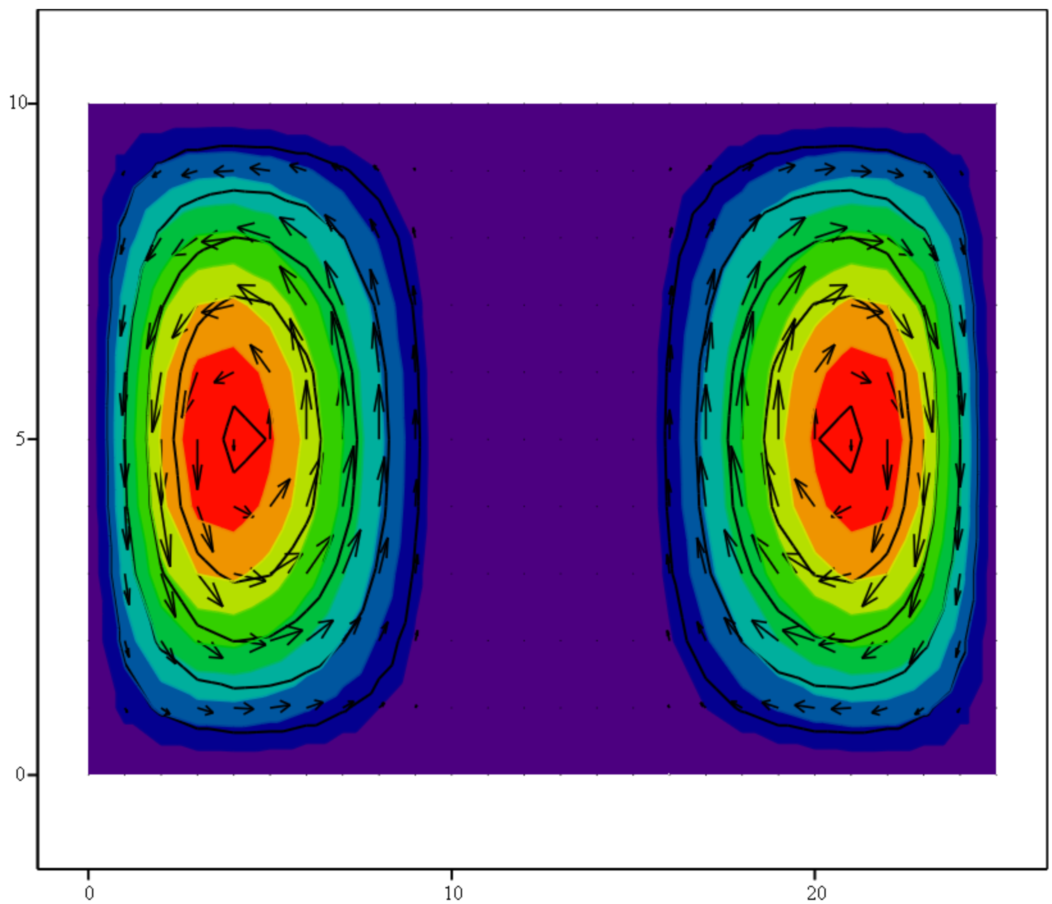
$zeroes_{3,jmax} := 0$

$CC := stack(reverse(J_z), zeroes, J_z)$

$AA := stack(reverse(Psi), zeroes, Psi)$

$DD := stack(reverse(J\phi(Psi)), zeroes, J\phi(Psi))$

$BB := stack(reverse(-J_r), zeroes, J_r)$



AA, (BB, CC), DD

References:
1. Y. Suzuki, Nuclear Fusion 14, 345 (1974).
2. K. Miyamoto, *Plasma Physics for Nuclear Fusion*, (MIT Press, Cambridge, Massachusetts) 1989. Chapter 7.6.