The harmonic oscillator: motion in a potential well

Consider a one dimensional electrostatic potential well: $\phi(x) := \alpha \cdot x^2$

The electric field is: $E(x) := -2 \cdot \alpha \cdot x$

The equations of motion for an ion are the two first order equations: $\frac{d}{dt}v(t) := \frac{q}{m} \cdot E(x)$ $\frac{d}{dt}x(t) := v(t)$

The little black boxes indicate that evaluation was disabled for these equations using the Format|Properties menu.

Define a 2-vector Z containing x in the first position and v in the second position.
Z will initially contain the starting values of x and v.

$Z := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Starting x value / Starting v value — Z is created using the Matrix menu for a 1 column, 2 row matrix.

Keep in mind that Z_0 is x and Z_1 is v.

We will let: $q := 1$ $m := 1$

Define α and E: $\alpha := 0.5$ $E(x) := -2 \cdot \alpha \cdot x$

The derivatives of x and v are:

$$DZ(t, Z) := \begin{pmatrix} Z_1 \\ \frac{q}{m} \cdot E(Z_0) \end{pmatrix}$$

DZ(t,Z) has the meaning:
dx/dt = v
dv/dt = qE/m

The derivatives DZ have arguments (t,Z) because the derivative is with respect to t and the derivatives are a function of Z.

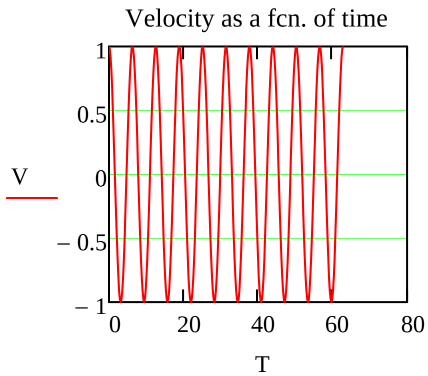
Use the Runge-Kutta routine **rkfixed** to integrate the equations. The frequency in radians/second of the oscillator is unity and we will follow it for 10 oscillations, thus the ending time is 20π. We will divide this interval into 300 points. The answers go in a matrix M.

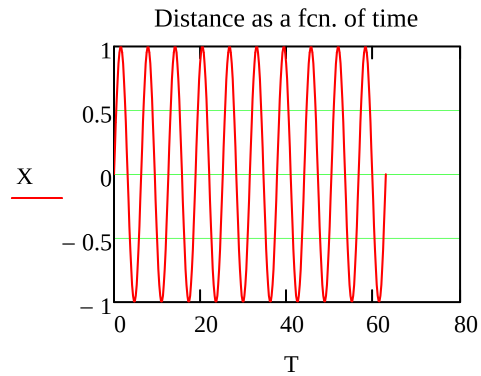
$npoints := 300$ $M := rkfixed(Z, 0, 20\pi, npoints, DZ)$

M will contain t in column zero, X (same as Z_0) in column 1, and V (same as Z_2) in column 2.
We will define new vectors T, X, and V as these columns of M and then make plots.

$T := M^{\langle 0 \rangle}$ $X := M^{\langle 1 \rangle}$ $V := M^{\langle 2 \rangle}$

These plots show that we indeed have a harmonic oscillator:





Phase error:

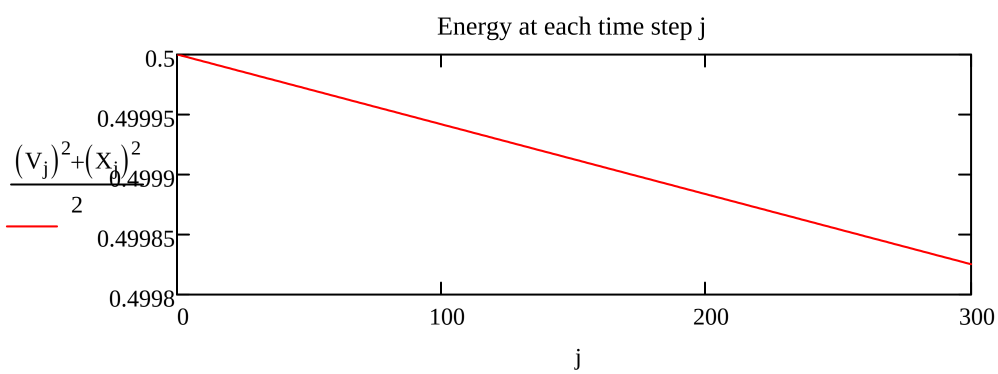
The "spring constant" (the force/distance) for our oscillator is 2α which is unity. Thus the period is 2π and our ending time of 20π should result x being zero at the end. Instead the x value is:

$X_{npoints} = -9.916 \times 10^{-4}$ Which is slightly less than zero. This is a small phase error.

Amplitude error:

Our harmonic oscillator has no damping, thus the sum of the potential energy (0.5x^2)and kinetic energies (0.5mv^2) should be a constant. Let's plot this sum and see what we get:

$j := 0 .. npoints$



$$\frac{(V_{npoints})^2 + (X_{npoints})^2}{(V_0)^2 + (X_0)^2} = 0.99965$$ The relative change in energy is very small.

Try it: How are the phase and amplitude errors affected by doubling npoints (halving the time step)? Is the error second order in the time step or fourth order?