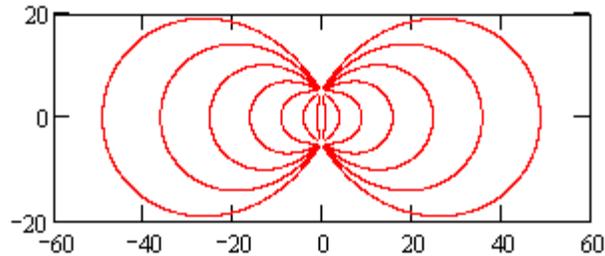# Plotting the field of an electrostatic dipole

What is an electric line of force? Is it the path a charged particle would follow if released in the electric field? No, because the particle will have inertia that will tend to carry it in the direction of the momentum it has accumulated. A tangent to the line of force drawn from a point on the line is parallel to the force at the point. This fact can be used to construct the line.



**The E field lines of a dipole.**

If we know how the E field varies in space, we can draw the line by "inching" forward in small increments ds that are drawn parallel to the force. In this exercise, we will draw the lines of force for an electrostatic dipole. The field for two point charges can be written down without solving Laplace's equation. This means that the components of the electric field E are given by simple formulas. We don't have to solve for them.
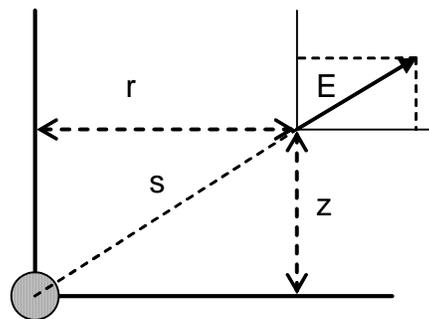
We will adopt cylindrical coordinates (r,z) with the z axis passing through the two point charges. A cylindrical coordinate system without the angular variable $\theta$ is appropriate because the dipolar field is not a function of $\theta$.

If the positive charge is at the origin, the electric field vector points away from the origin. The magnitude of the potential falls inversely with distance s from the charge and the magnitude of the field falls inversely with $s^2$.

We will begin by finding the field components $E_r$ and $E_z$ due to a charge at the origin. The drawing at right  is an aid to use in determining the components of E. The vector E is parallel to the line drawn from the location of the charge.  Define a constant C:



**The E vector (solid line) is parallel to a line drawn from the charge.**

$C := 1$      The magnitude of E is:

$$Emag(r, z) := \frac{C}{r^2 + z^2}$$

**Try it:** If the charge is Q, what is C in SI units?

From the similar triangles in the drawing, we see that radial component of E is the magnitude of E multiplied by r/s, where s is the distance.

$$s(r, z) := \sqrt{r^2 + z^2}$$      Then      $$Er(r, z) := Emag(r, z) \cdot \frac{r}{s(r, z)}$$

And similarly:      $$Ez(r, z) := Emag(r, z) \cdot \frac{z}{s(r, z)}$$

We will place our point charges on the z axis away from the origin. The positive charge will be at zpos and the negative point will be at zneg.

For the first graph of a point charge, let:     $\text{zpos} := 10$

To prepare a vector plot of E we will need a grid of points. We see above that division by zero can occur if the charge is at a grid point, so we will prepare a grid that avoids the origin. r will start at 0.5 and the grid point separation will be 1 unit:

| | | |
|---|---|---|
| $\text{jmax} := 10$ | $\text{kmax} := 20$ | The numbers of grid points. |
| $j := 0 .. \text{jmax}$ | $k := 0 .. \text{kmax}$ | The grid indices. |
| $\Delta r := 1$ | $\Delta z := 1$ | The grid point spacing. |
| $r_j := (j + 0.5)\cdot\Delta r$ | $z_k := k\cdot\Delta z$ | The r and z grid points. |

To obtain a plot of the vector E, we will choose the **vector field plot** from the Graph menu. The plot requires two matrices with the vector components $E_r$ and $E_z$.

These matrices will be:

$$ER_{j,k} := Er\left(r_j, z_k - \text{zpos}\right) \qquad EZ_{j,k} := Ez\left(r_j, z_k - \text{zpos}\right)$$
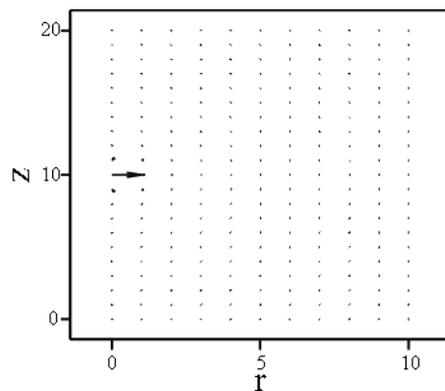


The j,k component of the matrices hold the values evaluated at $r_j$ and $z_k$. Note that the positive charge is at zpos.

**This vector plot** is not very helpful because the vector nearest the point charge is automatically scaled to be 1 unit long. All the other vectors are too small to be seen clearly because E falls so rapidly with distance.

The label command in the graph toolbox has been used to label the axes.

$$(ER, EZ)$$

Perhaps a nicer looking plot would be one with unit vectors at each point, with the unit vector pointing in the direction of E. The unit vector will be U and the components will be UR and UZ. The definitions of these vectors are:

$$UR_{j,k} := \frac{Er\left(r_j, z_k - \text{zpos}\right)}{\sqrt{Er\left(r_j, z_k - \text{zpos}\right)^2 + Ez\left(r_j, z_k - \text{zpos}\right)^2}} \quad UZ_{j,k} := \frac{Ez\left(r_j, z_k - \text{zpos}\right)}{\sqrt{Er\left(r_j, z_k - \text{zpos}\right)^2 + Ez\left(r_j, z_k - \text{zpos}\right)^2}}$$

The first definition defines UR as the radial component of E divided by the absolute value of E.

We can use symmetry to fill in the "left half" of the r,z plane.

Of course, r cannot be negative but if we create a negative r axis we will see what the field looks like when projected onto a plane containing the axis z.

To the left of this graph, we can position another graph with the vectors pointing to the left. That means for the left half that $E_r$ should be replaced with $-E_r$. We can create such a graph using the **reverse** command to create a mirror image of the matrices UR and UZ and then the **stack** command to place the matrices one above the other.

**This plot** is of the unit vectors.
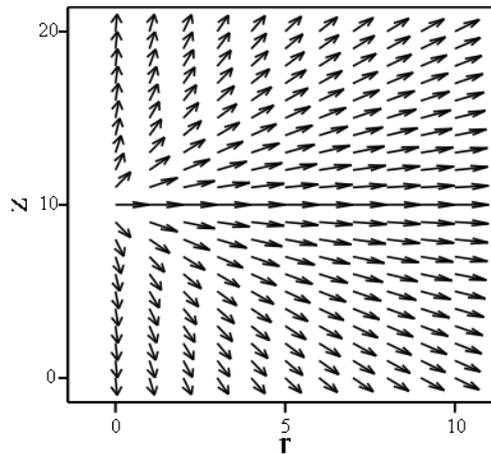


$$(UR, UZ)$$

The matrices contain values calculated at $r_j$, $z_k$. The rows are labelled with j and the columns are labeled with k. When you look at the <u>matrix</u>, z is increasing to the right and r is increasing downward. When you look at the <u>plot</u>, r is increasing to the right and z is increasing upward. Thus creating a mirror image of the plot above requires you to stop and think. Flipping the <u>plot</u> about the left vertical side is the same as flipping the <u>matrix</u> about the bottom edge.

Perform a simple test of the **reverse** command:
$$M := \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \text{reverse}(M) = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

$$UR2 := \text{stack}(\text{reverse}(-UR), UR)$$

$$UZ2 := \text{stack}(\text{reverse}(UZ), UZ)$$

For the left side of the **new graph below**, we have made the sign of $E_r$ negative and reversed the order of the rows.
**Try it:** Why was **stack** used, and not **augment**?
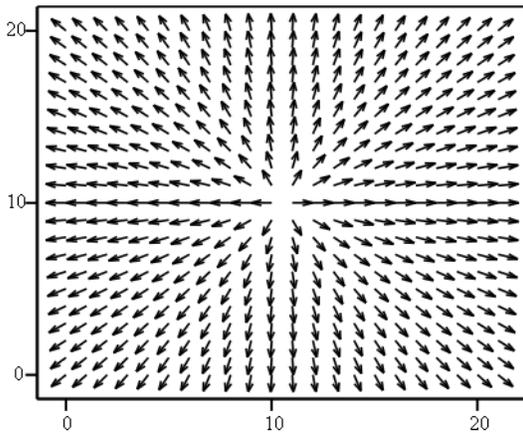


$$(UR2, UZ2)$$

If we had included r = 0 in the list of points, the new matrices would have two columns for r = 0 because the reversed matrix would also have r = 0. Using r = 0.5 for the first r value prevents this repetition of values from occurring.

### E field of a dipole (finally):

For the dipolar field, the point charges of the dipole will be at zpos = 5 and zneg = 15.

$zpos := 5$                    $zneg := 15$                    $z_{kmax} = 20$

The E fields of the two charges are obtained by summing the fields of the individual charges:

$$Er(r,z) := Emag(r, z - zpos) \cdot \frac{r}{s(r, z - zpos)} - Emag(r, z - zneg) \cdot \frac{r}{s(r, z - zneg)}$$

$$Ez(r,z) := Emag(r, z - zpos) \cdot \frac{z - zpos}{s(r, z - zpos)} - Emag(r, z - zneg) \cdot \frac{z - zneg}{s(r, z - zneg)}$$

The unit vectors are:     $UR_{j,k} := \dfrac{Er\left(r_j, z_k\right)}{\sqrt{Er\left(r_j, z_k\right)^2 + Ez\left(r_j, z_k\right)^2}}$     $UZ_{j,k} := \dfrac{Ez\left(r_j, z_k\right)}{\sqrt{Er\left(r_j, z_k\right)^2 + Ez\left(r_j, z_k\right)^2}}$

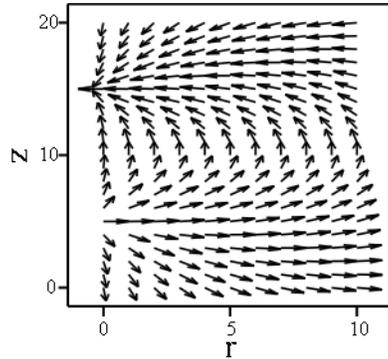**This plot** is the right half of the plot of the unit vectors pointing along E. The point charges are at r = 0, and at zpos and zneg defined above.

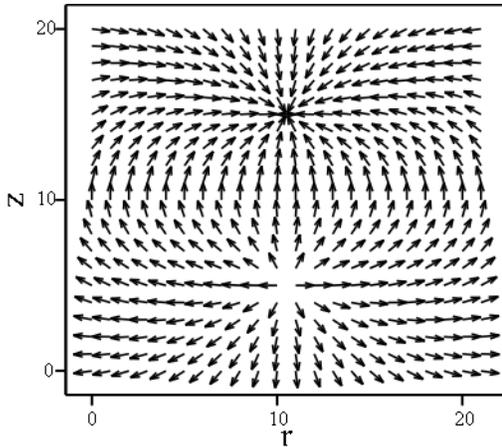To create the left half, we do the same **reverse** and **stack** as before:

$UR2 := stack(reverse(-UR), UR)$

$UZ2 := stack(reverse(UZ), UZ)$

**Plot of unit vectors for the dipole:**



(UR, UZ)



Why does the plot seem not to be up-down symmetric? That is because the heads of the arrows point away from the charge in the upper half plane and toward the charge in the lower half plane. In the upper half plane, the region around the charge is blacker because of the clustering of the arrow heads.

(UR2, UZ2)

Note that the horizontal axis is labelled with the grid point numbers rather than the r values.

## *Following a line of force:*

To follow a line of force, we simply "inch" forward in small increments ds, using the unit vectors UR and UZ as a guide. The amount forward in the r direction is UR ds and the amount forward in the z direction is UZ ds. This is because the scalars UR and UZ give the fraction of a unit vector in the r and z directions, respectively.

Algebraically

$$r(s) = \int_0^s dr = \int_0^s U_r \, ds \qquad\qquad z(s) = \int_0^s dz = \int_0^s U_z \, ds$$

The components of the unit vectors, written as functions of r and z, are:

$$Ur(r,z) := \frac{Er(r,z)}{\sqrt{Er(r,z)^2 + Ez(r,z)^2}} \qquad\qquad Uz(r,z) := \frac{Ez(r,z)}{\sqrt{Er(r,z)^2 + Ez(r,z)^2}}$$

We will create the trajectories of the lines of force by adding little increments dr and dz to a starting position.

We will use new variables R and Z to distinguish this part of the exercise from the earlier part.

$imax := 7$          Number of starting points for following the E field lines.

$i := 1 .. imax$       $rstart_i := 1.0 \cdot i^2$     Unevenly spaced starting positions.       $rstart_i =$

$ds := 0.1$          ds is the increment for "inching" forward.

| $rstart_i$ |
|---|
| 1 |
| 4 |
| 9 |
| 16 |
| 25 |
| 36 |
| 49 |

This **program loop** creates a list of points RZ that have the successive values of r and z for the trajectory of the line of force. Each trajectory begins on the midplane at z = 0. When the trajectory gets near to the point charge, the **while** loop stops and the calculations begin again with a new starting value for r. The value for ds is negative because the E field points down and we want our plotted lines to go up.

$$RZ := \begin{vmatrix} RZ \leftarrow \left( rstart_1 \quad 0 \right) \\ \text{for } i \in 1 .. imax \\ \quad \begin{vmatrix} R \leftarrow rstart_i \\ Z \leftarrow 0.5 \cdot z_{kmax} \\ \text{while } R^2 + (Z - zneg)^2 > 0.5 \\ \quad \begin{vmatrix} Rtemp \leftarrow R + Ur(R,Z) \cdot ds \\ Z \leftarrow Z + Uz(R,Z) \cdot ds \\ R \leftarrow Rtemp \\ RZ \leftarrow stack[RZ,(R \quad Z)] \end{vmatrix} \end{vmatrix} \\ RZ \end{vmatrix}$$

RZ has two columns, r values and z values, and the number of rows increases because the stack command is used after the new vaues are calculated to add the new values to the end.

**Try it:** In the program loop a temporary value of R is defined called Rtemp. What would have been the problem if Rtemp had not been used?

We want to plot both the upper and lower half planes so we do the same thing again with the ds values being negative so that the E field lines are followed from the midplane to the lower positive charge. The new RZ values are stacked onto the old RZ values and the new stack is called RZ2.

$$RZ2 := \begin{vmatrix} RZ2 \leftarrow RZ \\ \text{for } i \in 1 .. imax \\ \qquad \begin{vmatrix} R \leftarrow rstart_i \\ Z \leftarrow 0.5 \cdot z_{kmax} \\ \text{while } R^2 + (Z - zpos)^2 > 0.5 \\ \qquad \begin{vmatrix} Rtemp \leftarrow R + Ur(R,Z) \cdot (-ds) \\ Z \leftarrow Z + Uz(R,Z) \cdot (-ds) \\ R \leftarrow Rtemp \\ RZ2 \leftarrow stack[RZ2, (R \quad Z)] \end{vmatrix} \end{vmatrix} \\ RZ2 \end{vmatrix}$$

**This plot** is of the lines of force for point charges at zpos and zneg.     $zpos = 5$        $zneg = 15$
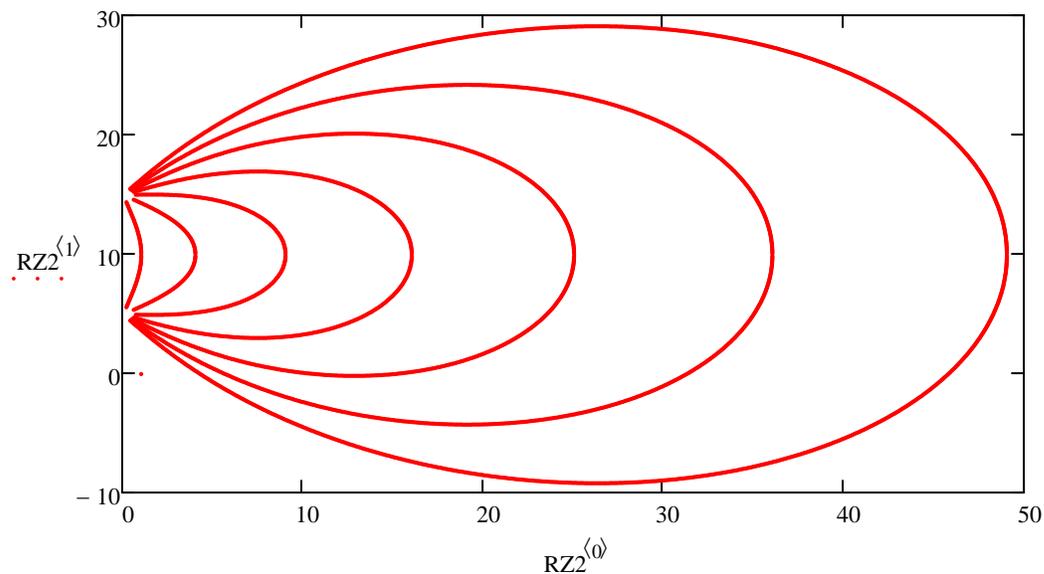


**Try it:** The plot above of the E field lines of the dipole does not show the left hand side that is the mirror image of what is plotted. Use the reverse and stack commands to create the left side of the plot and create a plot similar to the one at the top of p. 1 of this exercise.

**Try it:** Define a function $\Phi(r,z)$ that is the sum of the potentials of the two point charges, create a matrix with the values found at the grid points, and make a surface contour plot of $\Phi(r,z)$.

**Try it:** The method of integrating dr and dz to find the line of force is more accurate if the steps ds are smaller. Cut the figure above and paste it as a bitmap so that it does not change. Then repeat the calculations with ds made larger or smaller. Has ds been made sufficiently small that the results are independent of ds?

**Notes**
1. Accuracy: The method for following the lines is to use Euler's method to integrate the distances dr and dz. Euler's method is only accurate to first order. The plots could be made accurate to fourth order by using the Runge-Kutta integrator. However, the Mathcad **rkfixed** command integrates the line for a distance that must be specified, and we do not know in advance what distance s will take us from the midplane to the charge. If we want to use the Runge-Kutta integrator, we must resort to tricks. One trick would be to call **rkfixed** in a program loop and have it go forward only one step at a time. Then we could use a **while loop** to stop when we came to the end of the trajectory. Another trick would be specify an integration distance that is too long, and then to use an if(,,) statement to change ds to zero when the trajectory came to the charge. Then the Runge-Kutta integrator would continue at the charge but would not move the trajectory forward. The list of r,z points would be "padded" at the end with duplicates of the last set of points. In the Magnetostatics exercise "Plotting Field Lines," **rkfixed** is used to create the plots of the lines of B. For these plots, the integration does not have to be ended at a particular place.

2. The trajectories are point plots, not line plots: The trajectories have been plotted as points. The relatively small value of ds causes the points to merge visually into a line. If a line plot had been used, the line would jump from the end of the first trajectory to the beginning of the second because the trajectories are stacked. The stack command is a useful way to plot many lines on the same graph without giving each line a separate name that must be listed on the axes.

3. Division by zero: We defined the grid so that zero was not a grid point and this prevented division by zero in the calculations of E. An alternative method is to replace division by s with division by $(s+10^{-6})$, with some loss of accuracy.

4. Up-down symmetry: We could have avoided following the lines of force in the lower half plane by cutting and pasting below the mirror image of the upper half plane. In this case, the upper z axis should begin at 0.5 to avoid duplication of z = 0. The **reverse** command reverses rows and we must reverse columns so we must **transpose** the matrix, **reverse** it, and **transpose** again. Then **augment** is used to paste together the new reversed matrix and the old matrix. Further, some minus signs are needed in the vector fields because the vectors point toward the charge in one half plane and away from the charge in the other half plane. The commands are:

$$\text{UR3} := \text{augment}\left[ -\left(\text{reverse}\left(\text{UR2}^{\text{T}}\right)\right)^{\text{T}}, \text{UR2} \right] \qquad \text{UZ3} := \text{augment}\left[ -\left[\text{reverse}\left[(-\text{UZ2})^{\text{T}}\right]\right]^{\text{T}}, \text{UZ2} \right]$$

**Try it:** Create the plot of the lines of force in only the upper right quadrant only, and then use a sequence of **reverse, transpose, stack and augment** commands to create the full plot.

**References**
F. Hermann, H. Hauptmann, and M. Suleder, "Representations of electric and magnetic fields," Am. J. Phys. 68, 171 (2000). This paper discusses the difficulties encountered in trying to get the density of the lines of force to be proportional to the magnitude of E.