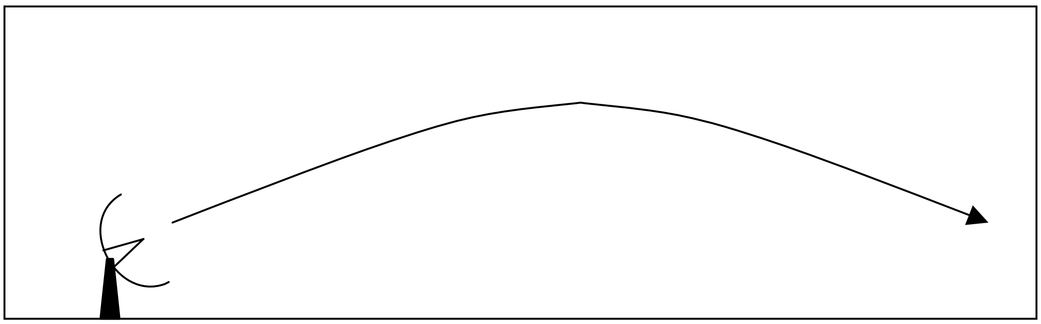
Ray tracing in an inhomogeneous medium

Radio waves can "bounce off" the ionosphere, particularly if they are incident at an angle. The index of refraction of the ionosphere changes gradually as a consequence of the increasing electron density. The group velocity of EM waves decreases with increasing altitude and the phase velocity increases. Reflection does not occur in the same way that reflection occurs in a mirror. Instead, the k vector gradually rotates toward the ground.



If we have a dispersion relation the in the form $\omega = \omega(\mathbf{x},\mathbf{k})$, we can take partial derivatives with respect to the components of vectors **x** and **k**, and use the constancy of ω to show that

$$\frac{d\vec{k}}{dt} = -\left.\frac{\partial \omega}{\partial \vec{x}}\right|_{\vec{k}}$$

$$\frac{d\vec{x}}{dt} = \left.\frac{\partial \omega}{\partial \vec{k}}\right|_{\vec{x}}$$

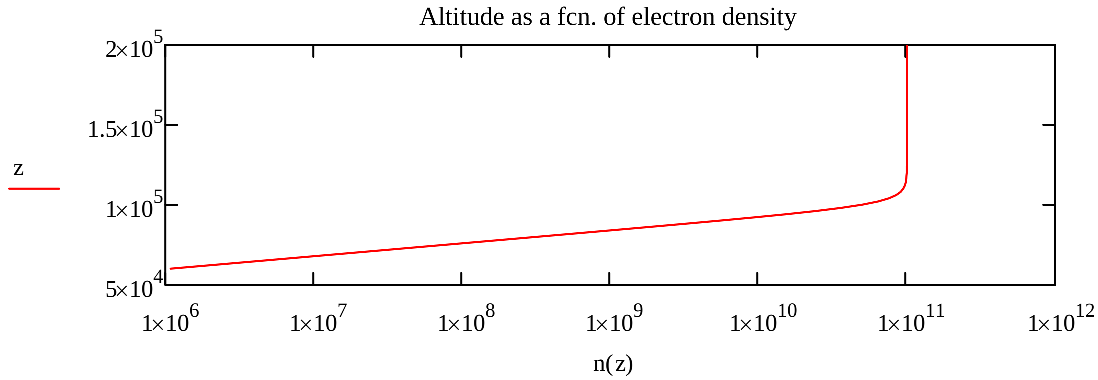
where the partial derivatives with respect to **k** and **x** are each taken with the other held constant. This approach is valid in the WKB approximation which requires that the properties of the medium change negligibly in a wavelength.
We will trace a ray bouncing off the ionosphere by using Runge-Kutta to advance a 6-vector Z containing the three components of **x** and **k**.

The electron density of the atmosphere is negligible below about 60 km and rises in the daytime to a value of about 10^{11} m^{-3} at 100 km. The density is lower at night and depends also upon solar activity.
We will express the electron concentration as an exponential of the form $1/(1+e^{\alpha z})$)

$$n(z) := \frac{10^{11}}{1 + e^{\frac{\left(10^5 - z\right)}{3500}}} \qquad m^{-3}$$

which is only good in the range

$$z := 60 \times 10^3, 62 \cdot 10^3 .. 200 \times 10^3 \qquad \text{meters}$$



At the knee in the curve, the electron density is not going up any more.

The dispersion relation:

The EM dispersion relation for the ionosopheric plasma, ignoring the magnetic field, is :

$$\omega^2 = \omega_p(z)^2 + c^2k^2$$

where $\omega_p(z)$ is :

$$\omega_p(z)^2 = n(z)q^2/\varepsilon_0 m$$

The method:

We will use Runge Kutta to advance the vector location of the ray, X, and the wavevector k.

$ORIGIN := 1$ This makes our subscripts 1,2..6 and not 0,1..5.

$$Z := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} X \\ Y \\ Z \\ k_x \\ k_y \\ k_z \end{matrix}$$

We will also need these recognizable physical constants:

$c := 3 \cdot 10^8$ $q := 1.6 \cdot 10^{-19}$

$m_e := 9.11 \cdot 10^{-31}$ $\varepsilon_0 := 8.854 \cdot 10^{-12}$

Now let's find and plot the ray

First pick a radian frequency: $\Omega := 1 \times 10^7$ This could be an AM radio channel.

Find the wavevector in vacuum: $k_{vac} := \frac{\Omega}{c}$

Pick an angle above the horizon for the pointing of the antenna $\theta := \frac{45}{90}\frac{\pi}{2}$

Pick starting X and V vectors $X := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $k := \begin{pmatrix} k_{vac} \cdot \cos(\theta) \\ 0 \\ k_{vac} \cdot \sin(\theta) \end{pmatrix}$ $k = \begin{pmatrix} 0.024 \\ 0 \\ 0.024 \end{pmatrix}$

We will stack these values and put them into our 6-vector z $Z := \text{stack}(X, k)$

The next two definitions let us recover X and k from the stack Z:

$X(Z) := \text{submatrix}(Z, 1, 3, 1, 1)$ $\quad k(Z) := \text{submatrix}(Z, 4, 6, 1, 1)$

We write the dispersion relation in the form $\omega = \omega(k(z),z)$

$$\omega(Z) := \sqrt{\frac{n(Z_3)\cdot q^2}{\varepsilon_0 \cdot m_e} + c^2\cdot(k(Z)\cdot k(Z))}$$

ω at the starting level for Z, which we will call Z0, is determined by only the ck term in the dispersion relation:

$Z0 := Z$ $\quad \omega(Z0) = 1 \times 10^7$ $\quad \sqrt{c^2\cdot k(Z)\cdot k(Z)} = 1 \times 10^7$

Only the z component of k changes, and for this we will need dn(z)/dz

$$\frac{dk_z}{dt} = -\frac{\partial\omega}{\partial z}\bigg|_{\vec{k}} = \frac{-q^2}{2\omega(z)\varepsilon_0 m_e}\frac{dn(z)}{dz}$$

We will place the derivative of n(z) in a new function called simply dn(z): $\quad dn(z) := \frac{d}{dz}n(z)$

This is done because Mathcad will not evaluate: $\quad \frac{d}{dZ_3}n(Z_3) =$

DY is the derivative of the 6-vector Z, and is defined using the WKB ray tracing equations above

$$DZ(t,Z) := \begin{pmatrix} \frac{c^2}{\omega(Z0)}\cdot Z_4 \\ \frac{c^2}{\omega(Z0)}\cdot Z_5 \\ \frac{c^2}{\omega(Z0)}\cdot Z_6 \\ 0 \\ 0 \\ \frac{-q^2}{2\cdot\omega(Z0)\cdot\varepsilon_0\cdot m_e}\cdot dn(Z_3) \end{pmatrix} \quad \begin{matrix} dx/dt \\ dy/dt \\ dz/dt \\ dk_x/dt \\ dk_y/dt \\ dk_z/dt \end{matrix}$$

How far should we carry the integration? We will guess that in going up and down to about 100 km altitude, the length travelled by the ray is the hypotenuse of two triangles with a height of 100 km.

$d := \frac{200000}{\sin(\theta)}$ $d = 2.828 \times 10^{5}$ The time to go this distance is $t := \frac{d}{c}$

We will integrate for a time t: $t = 9.428 \times 10^{-4}$ seconds

$npoints := 200$ We will guess that the ray changes a small amount at each point if we use 200 points.

$F := rkfixed(Z, 0, t, npoints, DZ)$

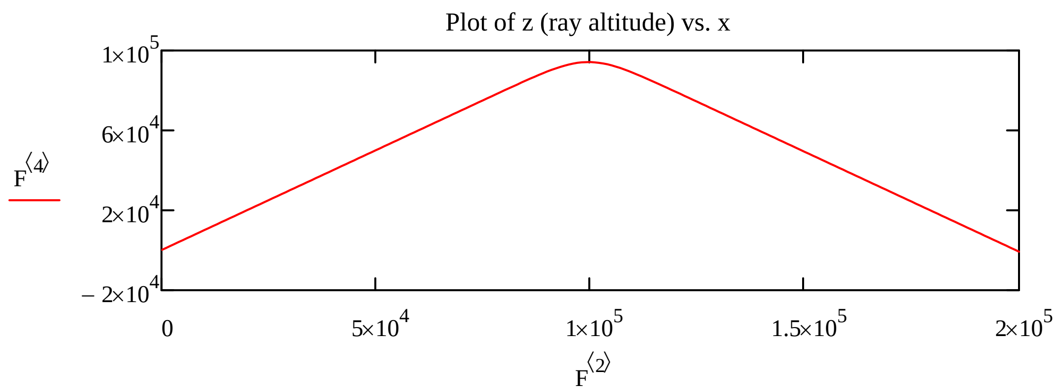
F =

	t	x	y	z	k_x	k_y	k_z
	1	2	3	4	5	6	7
1	$0.00 \cdot 10^{0}$	$0.00 \cdot 10^{0}$	$0.00 \cdot 10^{0}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
2	$4.71 \cdot 10^{-6}$	$10.00 \cdot 10^{2}$	$0.00 \cdot 10^{0}$	$10.00 \cdot 10^{2}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
3	$9.43 \cdot 10^{-6}$	$2.00 \cdot 10^{3}$	$0.00 \cdot 10^{0}$	$2.00 \cdot 10^{3}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
4	$1.41 \cdot 10^{-5}$	$3.00 \cdot 10^{3}$	$0.00 \cdot 10^{0}$	$3.00 \cdot 10^{3}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
5	$1.89 \cdot 10^{-5}$	$4.00 \cdot 10^{3}$	$0.00 \cdot 10^{0}$	$4.00 \cdot 10^{3}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
6	$2.36 \cdot 10^{-5}$	$5.00 \cdot 10^{3}$	$0.00 \cdot 10^{0}$	$5.00 \cdot 10^{3}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
7	$2.83 \cdot 10^{-5}$	$6.00 \cdot 10^{3}$	$0.00 \cdot 10^{0}$	$6.00 \cdot 10^{3}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
8	$3.30 \cdot 10^{-5}$	$7.00 \cdot 10^{3}$	$0.00 \cdot 10^{0}$	$7.00 \cdot 10^{3}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
9	$3.77 \cdot 10^{-5}$	$8.00 \cdot 10^{3}$	$0.00 \cdot 10^{0}$	$8.00 \cdot 10^{3}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
10	$4.24 \cdot 10^{-5}$	$9.00 \cdot 10^{3}$	$0.00 \cdot 10^{0}$	$9.00 \cdot 10^{3}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
11	$4.71 \cdot 10^{-5}$	$10.00 \cdot 10^{3}$	$0.00 \cdot 10^{0}$	$10.00 \cdot 10^{3}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
12	$5.19 \cdot 10^{-5}$	$1.10 \cdot 10^{4}$	$0.00 \cdot 10^{0}$	$1.10 \cdot 10^{4}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
13	$5.66 \cdot 10^{-5}$	$1.20 \cdot 10^{4}$	$0.00 \cdot 10^{0}$	$1.20 \cdot 10^{4}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
14	$6.13 \cdot 10^{-5}$	$1.30 \cdot 10^{4}$	$0.00 \cdot 10^{0}$	$1.30 \cdot 10^{4}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
15	$6.60 \cdot 10^{-5}$	$1.40 \cdot 10^{4}$	$0.00 \cdot 10^{0}$	$1.40 \cdot 10^{4}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	$2.36 \cdot 10^{-2}$
16	$7.07 \cdot 10^{-5}$	$1.50 \cdot 10^{4}$	$0.00 \cdot 10^{0}$	$1.50 \cdot 10^{4}$	$2.36 \cdot 10^{-2}$	$0.00 \cdot 10^{0}$	...

If we scroll down, we see that k_z begins to decrease at point 70 and reaches minus the initial k_z at point 130. So we have used enough points for the change per time step to be small.

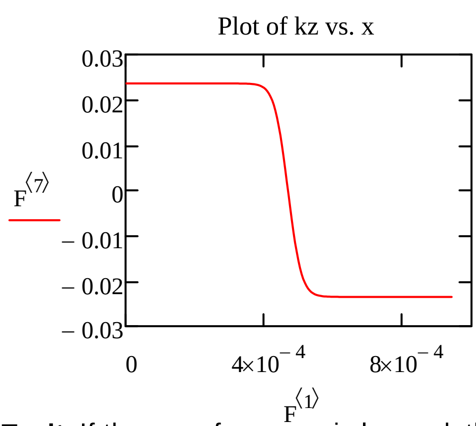
The vector of derivatives DZ has several entries of 0. We have written the problem out in three dimensions but only two are used. The reason for putting in the placeholders is so that we can easily change the problem to one that is truly three dimensional. For example, the earth's magnetic field would introduce an additional complication that we could include.

The wavelength is $\frac{2 \cdot \pi}{k_{vac}} = 188.496$ meters



$\max\left(F^{\langle 4\rangle}\right) = 9.412 \times 10^{4}$ The maximum value of z is the altitude of reflection.

Try it: Is the plasma frequency at this altitude more or less than the wave frequency?



In this plot we see that the z component of k reverses sign at the middle of the trajectory.

Try it: If the wave frequency is lowered, the reflection occurs at a lower altitude.
And if the frequency is raised too much, the Runge-Kutta fails (try it). Why?

Try it: Is the WKB approximation good for the wavelength calculated at the bottom of page 4?
What number on page 1 should the wavelength be compared with?

Notes:
1. The reference for the ray tracing equation is *Introduction to Plasma Physics*, R. J. Goldston and P. H. Rutherford (Institute of Physics, London, 1997), Ch. 15.3. And also *Plasma Waves*, D. G. Swanson (Academic Press, San Diego, 1989), Ch. 6.4.
2. The reference for the ionospheric density is: *Introduction to the Space Environment*, Thomas E. Tascione (Orbit Book Company, Florida, 1988).