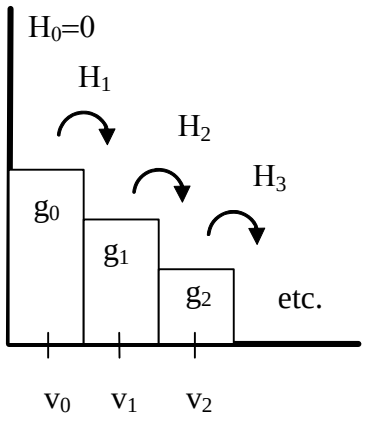
Fokker-Planck Equation

Collisions of electrons among themselves cause the electron velocity vectors to change in both length and direction. These changes are small and can be described as diffusion in velocity space. The Fokker-Planck (F-P) equation is the equation that describes diffusion arising from the accumulation of small changes. The F-P equation is similar to the diffusion equation and is based upon a continuity equation in velocity space:



$$\frac{\partial f(v)}{\partial t} + \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2\Gamma(v)\right] = 0.$$

where f(v) is the distribution function and Γ(v) is the flux of particles in velocity space. The equation is written in spherical velocity coordinates, because in this exercise we are only interested in the change in the length of the vector. The flux is given by:

$$v^2\Gamma(v) = -nYI(v)\left[f(v) + \frac{T}{mv}\frac{\partial f(v)}{\partial v} \right]$$

where Y is a constant, I(v) is a function that contains the velocity dependence of the collision frequency, T is the temperature in energy units, n is the number density, and m is the electron mass. The first term in the square brackets arises from the slowing of particles. For suprathermal particles, the speed after a collision is more likely to be a smaller speed, and this creates a flux of particles toward lower velocity that is proportional to f(v). The second term describes the diffusive flux in velocity space arising from collisions. The negative slope of f(v) makes the sign of the diffusive flux positive. The two terms for flux will cancel for a Maxwellian distribution colliding with itself. In this case the derivative of f(v) creates a factor -mv/T which results in cancellation of terms.

The Maxwellian distribution is: $$f(v) = n\left(\frac{m}{2\pi T}\right)^{3/2} \exp[-v^2 / v_t^2]$$

The velocity may be made dimensionless by the substitution u = v /v_t where: $v_t = \sqrt{2T/m}$

Every term of the equation contains an f(v), thus the equation may be simplified by dividing through by the factor $n(m/2\pi T)^{3/2}$, and using the substitution:

$$g(u) = \frac{f(v)}{n\left(m/2T\right)^{3/2}}$$

The constant Y is defined as:

$$Y = 4\pi\left(q^2 / 4\pi\varepsilon_0 m\right)^2 \ln\Lambda$$

where q is the electron charge, and ln Λ is the Coulomb logarithm.

The function I(v) is defined in terms of the error function ϕ(x) and its derivative ϕprime:

$$\phi(x) := \frac{2}{\sqrt{\pi}} \cdot \int_0^x e^{-y^2}\, dy \qquad \phi prime(x) := \frac{2}{\sqrt{\pi}} \cdot e^{-x^2} \qquad I(x) := \phi(x) - x \cdot \phi prime(x)$$

For these definitions, x is a dummy variable which will be replaced by the velocity u.
The dimensionless F-P equation is then:

$$\frac{\partial g(u,t)}{\partial \tau} = -\frac{1}{u^2}\frac{\partial}{\partial u}\left[u^2 \Gamma(u)\right] \qquad \text{and} \qquad u^2\Gamma(u) = H(u) = -I(u)\left[g(u) + \frac{1}{2u}\frac{\partial g(u)}{\partial u}\right]$$

where we have defined a collision frequency: $\nu = \frac{nY}{v_t^3}$

a scaled flux: $H(u) = u^2\Gamma(u)$ and a dimensionless time: $\tau = \nu\, t$

Finite difference form of the F-P equation

Finite-differencing of the continuity equation in dimensionless form gives:

$$\frac{g(u,\tau+\Delta t) - g(u,\tau)}{\Delta\tau} = -\frac{\left[(H(u+\Delta u/2,\tau) - H(u-\Delta u/2,\tau)\right]}{u^2\Delta u}$$

In finding the derivative of the scaled flux H, we have gone forward and backward half a step in u so that the derivative is centered at u. For g(u,τ+Δτ) we obtain:

$$g(u,\tau+\Delta\tau) = g(u,\tau) - \left(\frac{\Delta\tau}{\Delta u}\right)\frac{\left[H(u+\Delta u/2,\tau) - H(u-\Delta u/2,\tau)\right]}{u^2}$$

Finite differencing of the expression for the flux H(u) gives:

$$H(u+\Delta u/2,\tau) = -I(u+\Delta u/2)\left\{ g(u+\Delta u/2) + \frac{g(u+\Delta u,\tau) - g(u,\tau)}{2(u+\Delta u/2)\Delta u}\right\}$$

where we have centered the finite differencing of H at u + Δu/2 in order to keep H defined a the same points as in the continuity equation.

We cannot assign half-integer subscripts, so we will use integer subscripts with some of the functions being defined a half grid point from the position indicated by the subscript:

$u_m = (m+0.5)\Delta u$	The zeroth cell is centered a u = Δu/2.
$H_{m+1} = H(u_m + \Delta u/2)$	H_1 is the flux from cell 0 into cell 1.
$g_m = g(u_m, t)$	g_0 is the phase space density at u_0.
$I_{m+1} = I(u_m + \Delta u/2)$	(The cartoon at the top of p. 1 should clarify the relationships.)
$g(u_m + \Delta u/2) = \frac{1}{2}(g_{m+1} + g_m)$	

The reason for the last definition is that the finite difference form for H requires g(u) evaluated at both integer grid points and half integer grid points. However, g(u) is only stored at integer grid points. The half integer values are approximated by averaging the integer values on either side.

Replacing u by u_m in the finite difference equations, writing $g(u_m)$ as g_m, etc., and dropping the t dependence of H, we obtain:

$$g_m(\tau+\Delta\tau) = g_m(\tau) - \frac{\Delta\tau}{\Delta u}\frac{\left[H_{m+1} - H_m\right]}{{u_m}^2}$$

$$H_{m+1} = -I_{m+1}\left\{\tfrac{1}{2}[g_{m+1}+g_m] + \frac{g_{m+1}-g_m}{2(u_m+\Delta u/2)\Delta u}\right\}$$

which can be shifted down one index to obtain:

$$H_m = -I_m\left\{\tfrac{1}{2}[g_m+g_{m-1}] + \frac{g_m-g_{m-1}}{2(u_m-\Delta u/2)\Delta u}\right\}$$

As shown in the cartoon at the top of the first page, H_m depends upon the difference g_m - g_{m-1}, which appears in the equation for H_m above. In the continuity equation, g_m is increased by the flux entering, H_m, and is decreased by the flux leaving, H_{m+1}.

Two-step Lax-Wendroff method

Unfortunately the method outlined above is unstable. This can be fixed by using the Lax-Wendroff method that is correct to second order in time. In this method, the fluxes are calculated at time t and used to move forward half a time step. A temporary set of values, g^*_m, are defined at the half time step. These values of g_m are used to calculate the fluxes H^*_m at the time t + Δt/2. These time-centered fluxes are then used to advance g_m from t to t+Δt.

The half step for $g_m(t+\Delta t/2)$ is:
$$g^*_m = g_m(\tau) - \frac{\Delta\tau}{2\Delta u}\frac{\left[H_{m+1} - H_m\right]}{{u_m}^2}$$

H^*_m is calculated at the half time step:
$$H^*_m = -I_m\left\{\tfrac{1}{2}[g^*_m + g^*_{m-1}] + \frac{g^*_m - g^*_{m-1}}{2(u_m-\Delta u/2)\Delta u}\right\}$$

The full step for $g_m(t+\Delta t)$ is then taken using the fluxes calculated at t+Δt/2:

$$g_m(\tau+\Delta\tau) = g_m(\tau) - \frac{\Delta\tau}{\Delta u}\frac{\left[H^*_{m+1} - H^*_m\right]}{{u_m}^2}$$

A reference for this method is *Numerical Recipes in C* by W. H. Press, B. P. Flannery, S. A. Teukolsky and W. T. Vetterling (Cambridge University Press, Cambridge, 1988), Ch. 17.1.

Starting conditions for the F-P equation:

We will begin with a Maxwellian distribution having the tail cut off, and will watch the tail be replenished by diffusion of particles upward in velocity. This is slightly inconsistent. The function I(v) is calculated for a Maxwellian distribution colliding with itself. When we cut off the tail of the distribution, the distribution is no longer Maxwellian. However, the number of particles in the tail is small and the function I(v), which is obtained by integrating over f(v), is not significantly changed by cutting off the tail.

Define a velocity grid: $\Delta u := 0.1$ $mmax := 29$ $m := 0 .. mmax$ $u_m := \left(m + \frac{1}{2}\right)\cdot\Delta u$

The Maxwellian distribution g(u) in dimensionless units is: $g_m := \frac{1}{\sqrt{\pi^3}}\cdot e^{-(u_m)^2}$

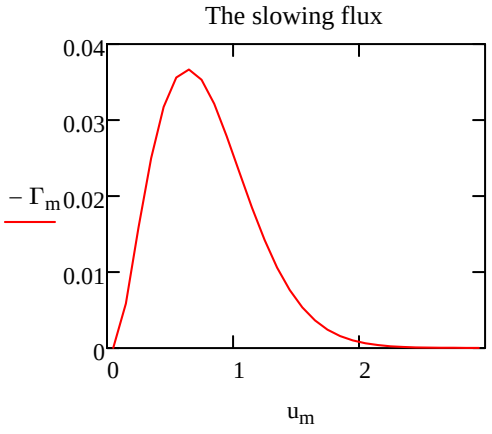
The special function I(u) evaluated on the half-grid: $I_m := I\left(u_m - \frac{\Delta u}{2}\right)$

The slowing-down flux:

Inspection of the F-P equation shows that the slowing flux is: $\Gamma_m := -\frac{I_m \cdot g_m}{(u_m)^2}$

For a Maxweillian, there is a diffusive flux that is equal in magnitude but opposite in sign.

Try it: Plot the diffusive flux on the same graph as the slowing flux and observe that they are equal in magnitude.

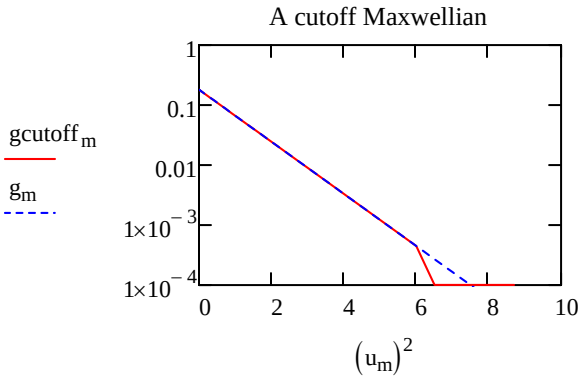


Filling-in of a missing tail:

The first problem we will solve is the filling-in of a distribution, gcutoff, that has a missing tail:

$$gcutoff_m := if\left[(u_m) < 2.5, \frac{e^{-(u_m)^2}}{\pi^{1.5}}, 10^{-4}\right]$$

The plot at right shows the distribution function gcutoff that we will start with. Also shown (dotted) is a full Maxwellian. The cutoff distribution will gradually fill in with particles as a consequence of diffusion in velocity space. The tail is set to 10^{-4} rather than zero because zero will not appear on a log plot.



Implementation of the two-step Lax-Wendroff method

We will follow the evolution of g for about 2 relaxation times. We will make our answer matrix M a function of the intial distribution, so that we can more easily calculate answers for other initial conditions. Our dimensionless time is counted in units of the relaxation time.

$tmax := 2$ Our time step must be small for stability: $\Delta\tau := 0.01$

The number of time steps will be: $jmax := \frac{tmax}{\Delta\tau}$ $jmax = 200$

$M(g) :=$ $M_{jmax,\,mmax} \leftarrow 0$ — Initialize M, the answer matrix of g values.

for $m \in 0 .. mmax$

$M_{0,m} \leftarrow g_m$

$M2_m \leftarrow g_m$ — M2 will hold the temporary values at the half time step.

$H_m \leftarrow 0$

for $j \in 1 .. jmax$

for $m \in 1 .. mmax$

$H_m \leftarrow -I_m \cdot \left[0.5 \cdot \left(M_{j-1,m} + M_{j-1,m-1} \right) + \frac{M_{j-1,m} - M_{j-1,m-1}}{2 \cdot \left(u_m - \frac{\Delta u}{2} \right) \cdot \Delta u} \right]$ — The fluxes at t.

for $m \in 0 .. mmax - 1$

$M2_m \leftarrow M_{j-1,m} - \frac{\Delta\tau \cdot \left(H_{m+1} - H_m \right)}{2 \cdot \left(u_m \right)^2 \cdot \Delta u}$ — This is a half step for the values M2 of g*.

for $m \in 1 .. mmax$

$H_m \leftarrow -I_m \cdot \left[0.5 \cdot \left(M2_m + M2_{m-1} \right) + \frac{M2_m - M2_{m-1}}{2 \cdot \left(u_m - \frac{\Delta u}{2} \right) \cdot \Delta u} \right]$ — The fluxes at t+Δt/2

for $m \in 0 .. mmax - 1$

$M_{j,m} \leftarrow M_{j-1,m} - \frac{\Delta\tau \cdot \left(H_{m+1} - H_m \right)}{\left(u_m \right)^2 \cdot \Delta u}$ — Now take a full step with the fluxes at t+Δt/2.

$M_{j,mmax} \leftarrow gcutoff_{mmax}$ — Preserve the boundary condition.

M

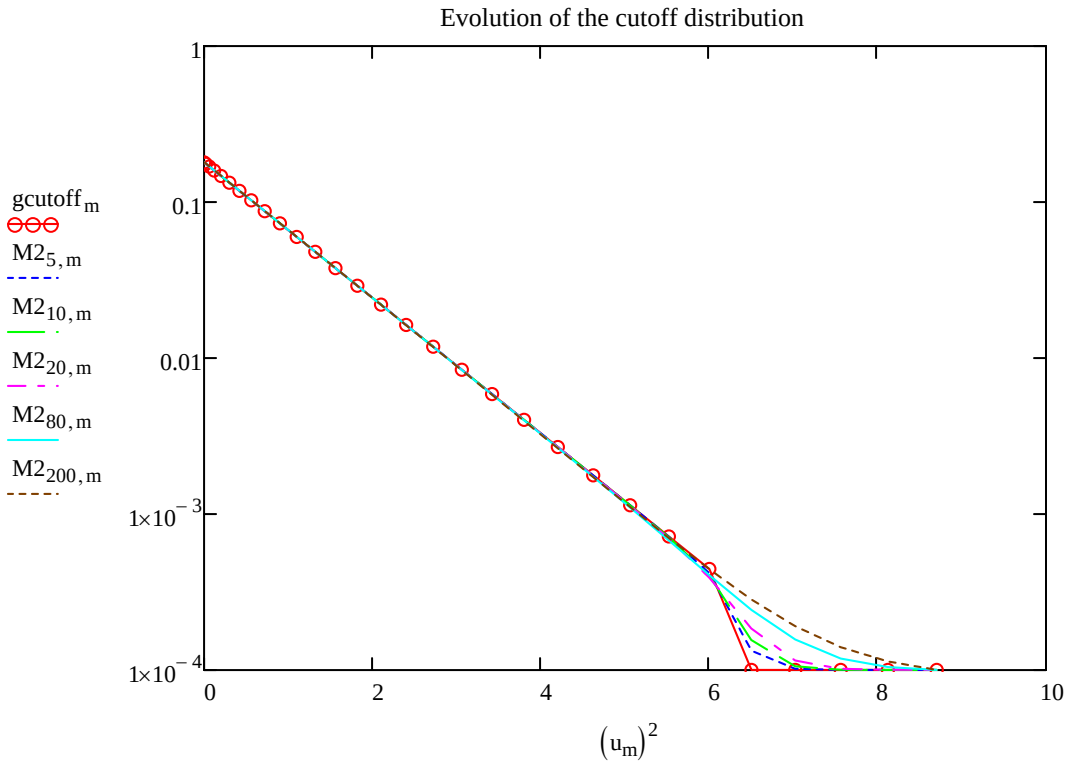
The flux H_0 is zero, so the **for loops** defining H_m start at 1, not zero.

The distribution g_m is set to zero at the boundary, m = mmax, and there is no need to update the value. Hence the for loops for calculating g_m end at mmax-1.

In the answer matrix M are the values of $g(u_m,t_j)$. The value of m increases to the right and the value of j increases going downward. The argument of M(g) is the starting distribution.

We have made M a function of g so that we can start with any distribution.
First, we will find and plot the evolution of the cutoff distribution.

Calculate the values of $g(u_m,t_j)$ and place them in matrix M2 for plotting: $M2 := M(gcutoff)$



Inspection shows that the tail indeed fills in. The bulk of the distribution remains Maxwellian because self-collisions do not alter a Maxwellian. There is a small change in the bulk Maxwellian because of particle conservation. The addition of particles to the tail resulted in a small reduction in the number of particles in the bulk of the distribution.

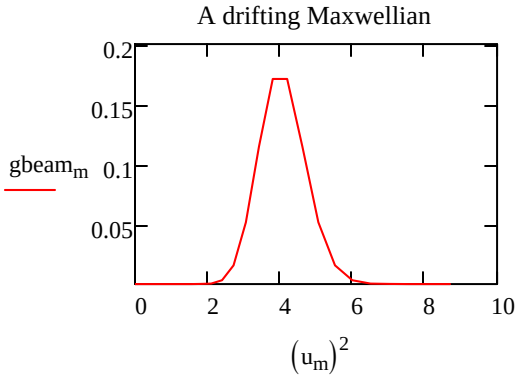
Slowing of a beam

Fusion energy devices are often heated by beams of energetic particles. The beam particles lose energy to the bulk plasma as they slow down. The second example is the evolution of a slowing beam. Begin by defining the initial beam distribution function:

$$gbeam_m := \frac{\exp\left[-20\left(2 - u_m\right)^2\right]}{\pi^{1.5}}$$

This is the distribution that we will assign to our beam.

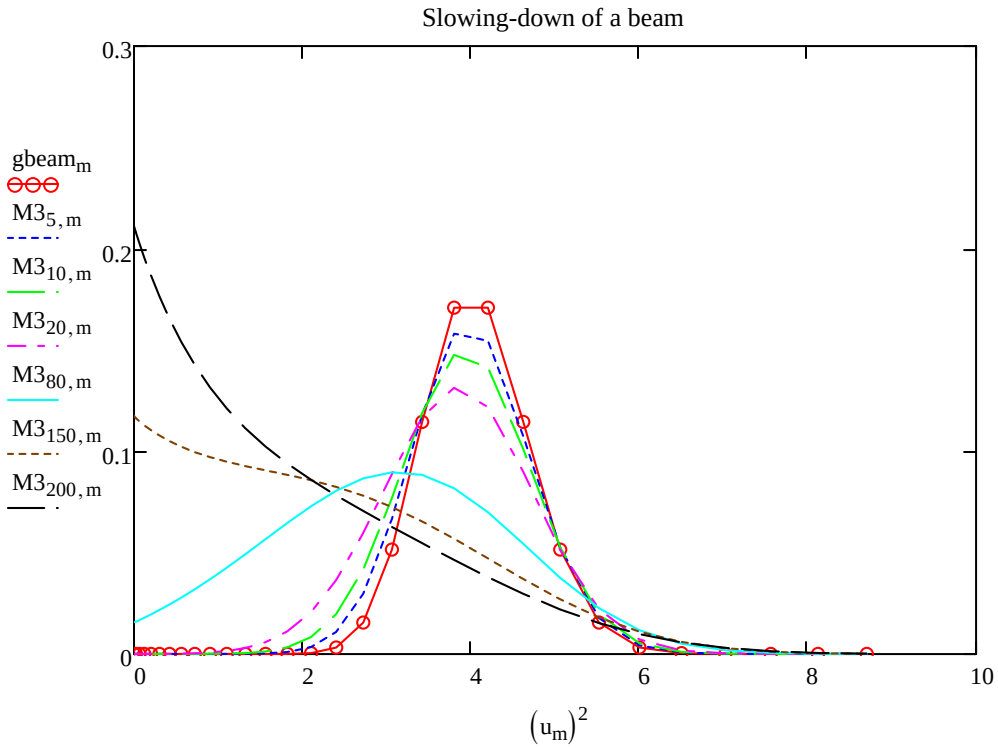
This distribution is a Maxwellian that is centered at velocity = 2. Our problem is expressed in spherical velocity coordinates, so this distribution is a spherical shell in velocity space.



Calculate a new answer matrix:

$M3 := M(gbeam)$

We place the answer matrix in a temporary matrix M3 for plotting.



This plot is linear rather than logarithmic. The beam indeed slows down and at late times begins to resemble a Maxwellian.

Particle conservation:

The number density of particles in a distribution is: $n = \int_0^\infty g(u) 4\pi u^2 du$

This can be converted to a sum: $n(g) := \sum_m \left[g_m \cdot 4 \cdot \pi \cdot \left(u_m\right)^2 \cdot \Delta u \right]$

The normalized number density for the Maxwelllian is: $n(g) = 1$

Let's compare the initial number of particles with the final number of particles for the beam slowing problem.

The initial density is: $n(gbeam) = 3.6$

The final distribution plotted is row 200 of matrix M3: $gfinal_m := M3_{200,m}$

The final density is: $n(gfinal) = 3.595$

The final density is very near to the initial density: $\frac{n(gfinal)}{n(gbeam)} = 0.999$

Some particles were lost from the distribution at the right boundary.

Notes:
1. Stability requires that very small time steps (0.01) be made. The time step can be made an order of magnitude larger (0.1) if a "fully implicit" method is used, such as the "Crank-Nicholson" method. These methods require inverting a tridiagonal matrix, which is a bit beyond the scope of these exercises. See *Numerical Recipes in C*, by Press, Flannery, Teukolsky and Vetterling, Ch. 17.2.
2. The Fokker-Planck equation is discussed in *The Particle Kinetics of Plasmas,* by I.P. Shkarofsky, T. W. Johnston and M. P. Bachynski (Addison-Wesley, New York, 1966).